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COMMON FIXED POINTS OF PSEUDO COMPATIBLE MAPPINGS IN FUZZY METRIC SPACES

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Abstract In the present paper, the concepts of pseudo compatible, g-reciprocal continuous and occasionally weakly compatible mappings have been used in the context of existence of common fixed points for a pair of mappings. Results proved un this paper do not require the conditions of continuity of maps and closedness of any range. Hence, results proved in this paper generalize many known related common fixed point theorems in the literature in the setting of Menger, Fuzzy, Intuitionistic fuzzy and Modified fuzzy metric spaces.

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1. Introduction

After the appearance of the notion of fuzzy set [13], the literature in this area has risen very quickly because of its interesting applications in applied sciences One of the most interesting research topics in fuzzy topology is to find an appropriate definition of fuzzy metric space. Many authors have considered this problem and have introduced it in different ways. Inspired by Menger spaces, Kramosil and Michalek [4] generalized the concept of probabilistic metric space to the fuzzy framework. Thereafter, many authors (see e.g.,[2],[3],[4-9]) established fuzzy version of most of the classical metric common fixed point theorems.

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In the present paper, the concepts of pseudo compatible, g-reciprocal continuous and occasionally weakly compatible mappings have been used in the context of existence of common fixed points for a pair of mappings.

2. Preliminaries

Definition 2.1 (13). A fuzzy set A in a universe X is an object $A = \{(x, \mu_A(x)) : x \in X\}$ where $\mu_A : X \to [0, 1]$ is the membership function or grade of membership of the A.

Definition 2.2 (12). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if $*$ satisfies the following conditions: for all $a, b, c, d \in [0, 1]$,
(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$;
(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$.

Definition 2.3 (4). The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,
(i) $M(x, y, 0) = 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.
The function $M(x, y, t)$ denote the degree of nearness between x and y w.r.t. t.

Remark 2.4 (3). The mapping $M(x, y, .)$ is non-decreasing for all $x, y \in X$ in fuzzy metric space $(X, M, *)$.

Definition 2.5 (2). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be
(i) convergent to a point $x \in X$, if for each $\epsilon \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \epsilon$, for all $n \geq n_0$.
(ii) Cauchy sequence if for each $\epsilon \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$, for all $m, n \geq n_0$.

Definition 2.6 (2). A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

Definition 2.7 (9). A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be compatible if $lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = z$ for some $z \in X$.

A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = z$ but $lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or non-existence for at least one $t > 0$.

Definition 2.8 (1). A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be occasionally weakly compatible (owc) if there exists a point $x \in X$ which is a coincidence point of f and g at which f and g are compatible.

Definition 2.9 (10). A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be reciprocally continuous if $lim_{n\to\infty} fgx_n = fz$ and $lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.10 (12)**.** Let f and g be self mappings of a fuzzy metric space $(X, M, *)$. Then for a sequence $\{y_n\}$ in X satisfying $lim_{n\to\infty} fy_n = lim_{n\to\infty} gy_n = t$, a sequence $\{z_n\}$ will be called an associated sequence if $fy_n = gz_n$ or $gy_n = fz_n$ and $lim_{n\to\infty} fz_n = lim_{n\to\infty} gz_n$.

Definition 2.11 (12)**.** A pair (f, g) of self-mappings of a fuzzy metric space $(X, M, *)$ is said to be pseudo compatible iff whenever the set of sequence $\{x_n\}$ satisfying $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n$ is nonempty, there exists a sequence $\{y_n\}$ in X satisfying $lim_{n\to\infty} fy_n = lim_{n\to\infty} gy_n = t$, $lim_{n\to\infty} M(fgy_n, gfy_n, t) = 1$, and $lim_{n\to\infty} M(fgz_n, gfz_n, t) = 1$, for any associated sequence $\{z_n\}$ of $\{y_n\}$.

Definition 2.12 (13)**.** Two self mappings f and g be self mappings of a fuzzy metric space $(X, M, *)$ are called g-reciprocally continuous iff $lim_{n\to\infty} ffx_n = ft$ and $lim_{n\to\infty} gfx_n = gt$ whenever sequence $\{x_n\}$ is a sequence such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Lemma 2.13 (9)**.** *Let $(X, M, *)$ be a fuzzy metric space. If there exists a constant $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X, t > 0$, then $x = y$.*

3. Main Results

Theorem 3.1. *Let f and g be g-reciprocally continuous self mappings of a complete fuzzy metric space $(X, M, *)$ such that:*
(i) $f(X) \subseteq g(X)$;
(ii) $M(fx, fy, qt) \geq M(gx, gy, t)$ *where* $q \in (0, 1)$.

If f and g are pseudo compatible then f and g have a unique common fixed point.

Proof. Let x_0 be any point in X. Then since $f(X) \subseteq g(X)$, there exists a sequence of points x_n in X such that $fx_0 = gx_1, fx_1 = gx_2, ..., fx_n = gx_{n+1}$.

Also define a sequence $\{s_n\}$ in X as $s_n = fx_n = gx_{n+1}$ for all $n = 0, 1, 2, ...$

We prove that $\{s_n\}$ is a Cauchy sequence. Using (ii) we obtain

$$M(s_n, s_{n+1}, qt) = M(fx_n, fx_{n+1}, qt) \geq M(gx_n, gx_{n+1}, t)$$
$$M(s_n, s_{n+1}, qt) \geq M(s_{n-1}, s_n, t).$$

Therefore we can write

$$M(s_2, s_3, qt) \geq M(s_1, s_2, t),$$
$$M(s_1, s_2, qt) \geq M(s_0, s_1, t),$$

which implies that

$$M(s_1, s_2, t) \geq M(s_0, s_1, \frac{t}{q}).$$

We have

$$M(s_2, s_3, t) \geq M(s_0, s_1, \frac{t}{q^2}).$$

By simple induction with condition (ii) we have for all $t > 0$,

(3.1)
$$M(s_n, s_{n+1}, t) \geq M(s_0, s_1, \frac{t}{q^n}).$$

Also for every integer $p > 0$, we get

$$M(s_n, s_{n+p}, t) \geq M(s_n, s_{n+1}, \frac{t}{p}) * M(s_{n+1}, s_{n+2}, \frac{t}{p}) * ... * M(s_{n+p-1}, s_{n+p}, \frac{t}{p}).$$

Using (3.1) we will get

$$M(s_n, s_{n+p}, t) \geq M(s_0, s_1, \frac{t}{pq^n}) * M(s_0, s_1, \frac{t}{pq^{n+1}}) * ... * M(s_0, s_1, \frac{t}{pq^{n+p-1}}),$$

letting $n \to \infty$ we obtain

$$lim_{n\to\infty} M(s_n, s_{n+p}, t) \geq 1 * 1 * ... * 1 = 1.$$

Hence $\{s_n\}$ is a Cauchy sequence. Since X is complete therefore there exists a point $u \in X$ such that sequence $\{s_n\}$ is convergent.

Therefore, pseudo compatibility of f and g implies there exists $\{y_n\}$ such that

$$lim_{n\to\infty} fy_n = lim_{n\to\infty} gy_n = u$$

and

$$lim_{n\to\infty} M(fgy_n, gfy_n, t) = 1.$$

Since $f(X) \subseteq g(X)$, for each y_n there exists a $z_n \in X$ such that $fy_n = gz_n$ for all n. This implies

$$lim_{n\to\infty} fy_n = lim_{n\to\infty} gy_n = lim_{n\to\infty} gz_n = u.$$

By virtue of this and using (ii) we obtain $fz_n \to u$ which in turns implies that $\{y_n\}$ and $\{z_n\}$ are associated sequences and $lim_{n\to\infty} M(fgz_n, gfz_n, t) = 1$. Now g-reciprocal continuity of f and g implies that $lim_{n\to\infty} ffy_n = fu$, and $lim_{n\to\infty} gfy_n = lim_{n\to\infty} ggz_n = u$. Similarly $lim_{n\to\infty} ffz_n = fu$ and $lim_{n\to\infty} gfz_n = gu$.

Pseudo compatibility of f and g implies $lim_{n\to\infty} fgy_n = gu$.

$lim_{n\to\infty} fgy_n = lim_{n\to\infty} fgz_n = gu$.

Hence $lim_{n\to\infty} gfz_n = gu$. Using (ii) we get

$$M(fu, fgz_n, qt) \geq M(gu, ggz_n, t),$$

letting $n \to \infty$, we get

$$M(fu, gu, qt) \geq M(gu, gu, t).$$

By using Lemma 2.13, we have $fu = gu$.

Again using (ii) we get

$$M(fu, fz_n, qt) \geq M(gu, gz_n, t),$$

letting $n \to \infty$, we get

$$M(fu, u, qt) \geq M(gu, u, t) = M(fu, u, t)$$

implies $fu = u$.

Therefore u is a fixed point. Uniqueness of fixed point easily follows by using (ii). Hence the result. ∎

Theorem 3.2. *Let $(X, M, *)$ be a fuzzy metric space. If f and g are occasionally weakly compatible self mappings of X satisfying:*
(i) $M(fx, ffx, t) \neq min\{M(fx, gfx, t), M(ffx, gfx, t)\}$, whenever $fx \neq ffx$.
Then f and g have a common fixed point.

Proof. Since f and g are owc, there exists a point $u \in X$ such that $fu = gu$ and $fgu = gfu$ which in turns gives $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, then using (i) we get

$$M(fu, ffu, t) \neq min\{M(fu, gfu, t), M(ffu, gfu, t)\},$$

$$M(fu, ffu, t) \neq min\{M(fu, gfu, t)\},$$

implies

$$M(fu, ffu, t) \neq M(fu, ffu, t),$$

a contradiction implies that $fu = ffu = fgu = gfu$. Hence $fu = gu$ is a common fixed point of f and g. ∎

Theorem 3.3. *Let f and g be g-reciprocally continuous non-compatible self mappings of a fuzzy metric space $(X, M, *)$ such that:*
(i) $f(x) \subseteq g(X)$;
(ii) $M(fx, fy, t) \geq min\{M(gx, gy, t), \frac{k[M(fx,gx,t)*M(fy,gy,t)]}{2}, \frac{[M(fx,gy,t)*M(fy,gx,t)]}{2}\}$ where $1 \leq k \leq 2$;
(iii) $M(x, fx, t) \neq min\{M(x, gx, t), M(fx, gx, t)\}$, whenever $fx \neq x$.
If f and g are pseudo compatible then f and g have a unique common fixed point.

Proof. Since f and g are non-compatible maps, there exists a sequence $\{x_n\}$ in X such that $fx_n \to z$ and $fx_n \to z$ for some $z \in X$ but either $lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exists. Pseudo compatibility of f and g implies there exists a sequence $\{y_n\}$ such that $lim_{n\to\infty} fy_n = lim_{n\to\infty} gy_n = u$ and $lim_{n\to\infty} M(fgy_n, gfy_n, t) = 1$. Since $f(X) \subseteq g(X)$, for each y_n, there exists a z_n in X such that $fy_n = gz_n$ for all n. This implies that $lim_{n\to\infty} fy_n = lim_{n\to\infty} gy_n = lim_{n\to\infty} gz_n = u$ using this and equation (ii) we obtain $fz_n \to u$. Therefore $\{y_n\}$ and $\{z_n\}$ are associated sequences and consequently $lim_{n\to\infty} M(fgz_n, gfz_n, t) = 1$. Further g-reciprocal continuity of f and g implies that $ffy_n \to fu$ and $gfy_n = ggz_n \to gu$. Similarly $ffz_n \to fu$ and $gfz_n \to gu$. In view of this and pseudo compatibility, we show that f and g we get $fgy_n \to gu$ and $fgz_n = ffy_n \to gu$. Hence $fu = gu$. Again, if $u \neq fu$, then using (iii) we obtain

$$M(u, fu, t) \neq min\{M(u, gu, t), M(fu, gu, t)\} = M(u, fu, t),$$

a contradiction. Hence $fu = gu = u$ and u is a common fixed point of f and g.

Uniqueness of the common fixed point follows easily.

We now show that f and g are discontinuous at common fixed point u. If possible, suppose f is continuous at u. Then consider the sequence $\{x_n\}$ of the present theorem, we get $ffx_n \to fu = u$ and $fgx_n \to gu = u$. Now g-reciprocal continuity implies that $lim_{n\to\infty} ffx_n = fu = u$ and $lim_{n\to\infty} gfx_n = gu = u$. This, further yields $lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, which contradicts that either $lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exists. Hence f is discontinuous at the fixed point.

Next, suppose that g is continuous at u. Then, for the sequence $\{x_n\}$, we get $gfx_n \to gu = u$ and $ggx_n \to gu = u$. In view of these limits, the inequality
$M(fu, fgx_n, t) \geq$
$min\{M(gu, ggx_n, t), \frac{k[M(fu,gu,t)*M(fgx_n,ggx_n,t)]}{2}, \frac{[M(fu,ggx_n,t)*M(fgx_n,gu,t)]}{2}\}$
a contradiction implies $fgx_n \to fu = u$ but $fgx_n \to u$ and $gfx_n \to u$ contradicts the fact that either $lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exists. Thus f and g both are discontinuous at the fixed point. Hence the result. ∎

4. Conclusion

The results presented in this paper generalized various known fixed point theorems in the literature in setting of Menger, Fuzzy, Intuitionistic fuzzy and Modified fuzzy metric spaces since results presented in this paper does not require: closedness of ranges; continuity of any self maps.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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