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The existence of best proximity points and fixed points for new nonlinear mappings on quasiordered metric spaces

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Abstract In this paper, we establish some new existence and convergence theorems of iterates of best proximity points on quasiordered metric spaces. Some applications to the fixed point theory are also given. Our results generalize and improve some known results in the literature.

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1. Introduction and preliminaries

The study of fixed point theory is an important subject of nonlinear analysis and is frequently applied to nonlinear integral equations and differential equations. For a nonempty subset D of a metric space X and a non-self mapping $T : D \to X$, it is known that the equation $Tx = x$ does not necessarily have a solution. We often turn to investigate the best approximation of the existence of solutions if the equation $Tx = x$ has no solution. In the last decade, a number of generalizations in various directions on the existence and uniqueness of a best proximity point were investigated by several authors; see, e.g., [1-23] and references therein.

Let A and B be nonempty subsets of a nonempty set E. A mapping $S : A \cup B \to A \cup B$ is called *cyclic* if $S(A) \subset B$ and $S(B) \subset A$. Let (X, d) be a metric space and $T : A \cup B \to A \cup B$ be a selfmapping. For any nonempty subsets A and B of X, let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

A point $x \in A \cup B$ is called to be a best proximity point for T if $d(x, Tx) = dist(A, B)$.

In 2003, Kirk, Srinavasan and Veeramani [13] extended and generalized the Banach contraction principle that introduced cyclic mappings and best proximity points. In 2006, Eldered and Veeramani [9] established some results about best proximity points of cyclic contraction mappings.

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Definition 1.1. [9] Let A and B be nonempty closed subsets of a complete metric space (X, d). A cyclic map $T\colon A \cup B \to A \cup B$ is called a *cyclic contraction* if for some $\alpha \in (0, 1)$, the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)dist(A, B)$$

holds for all $x \in A$ and $y \in B$.

Theorem 1.1. **[9, Proposition 3.2]** Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map, $x_1 \in A$ and define $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.

Let X be a nonempty set and $\preccurlyeq$ a quasiorder (preorder or pseudo-order, i.e., a reflexive and transitive relation) on X. Then $(X, \preccurlyeq)$ is called a *quasiordered set*. If (X, d) is a metric space with a quasiorder $\preccurlyeq$, we call it a *quasiordered metric space* $(X, d, \preccurlyeq)$ for short. Let $(X, \preccurlyeq)$ be a quasiorder set and $T : X \to X$ be a selfmapping. We say that T is $\preccurlyeq$-*nonincreasing* if $x, y \in X$ with $x \preccurlyeq y$ implies $Ty \preccurlyeq Tx$ or, equivalently, $Tx \succcurlyeq Ty$. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the sets of positive integers and real numbers, respectively.

In 2011, Derafshpour, Rezapour and Shahzad [6] proved the following convergence theorem on ordered metric spaces.

Theorem 1.2. **[6, Theorem 2.2]** *Let $(X, d, \preccurlyeq)$ be an ordered metric space, $A, B \in 2^X$ and T a $\preccurlyeq$-nonincreasing selfmapping on $A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Suppose that there exists $x_0 \in A$ such that $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$ and*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(dist(A, B))$$

for all $x \in A$ and $y \in B$ with $x \preccurlyeq y$, where $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing function. If $x_{n+1} = Tx_n$ and $d_n = d(x_{n+1}, x_n)$ for all $n \geq 0$, then $d_n \to dist(A, B)$ as $n \to \infty$.

In this paper, we establish some new existence and convergence theorems of iterates of best proximity points on quasiordered metric spaces. Some applications to fixed point theory are also given. Our results generalize and improve Derafshpour-Rezapour-Shahzad's convergence theorem and include some known results in the literature as special cases.

2. Some best proximity point theorems on quasiordered metric spaces

In this section, we first establish the following convergence theorem for the best proximity points on quasiordered metric spaces which is one of the main results in this paper.

Theorem 2.1. *Let $(X, d, \preccurlyeq)$ be a quasiordered metric space, A and B be nonempty subset of X and T be a cyclic $\preccurlyeq$–nonincreasing selfmapping on $A \cup B$. Suppose that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \mu(\min\{d(Tx, y), d(x, Ty)\}) + \varphi(dist(A, B)) \quad (2.1)$$

for all $x \in A$ and $y \in B$ with $x \preccurlyeq y$, where $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing function and $\mu : [0, \infty) \to \mathbb{R}$ is a function satisfying $\mu(0) \leq 0$. Suppose that there *exists* $x_0 \in A$ *such that* $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$. *Define an iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_{n+1} = Tx_n$ *for* $n \in \mathbb{N}\cup\{0\}$. *Then the following hold.*

(a) $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})) + \varphi(dist(A, B))$ *for all* $n \in \mathbb{N}\cup\{0\}$;

(b) $\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.

Moreover, if $d(Tx, Ty) \leq d(x, y)$ *for any* $x \in A$ *and* $y \in B$ *and* $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ *has a convergent subsequence in* A, *then there exists* $v \in A$ *such that* $d(v, Tv) = dist(A, B)$.

Proof. Since $x_0 \in A$, $T(A) \subseteq B$ and $T(B) \subseteq A$, we have $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N}\cup\{0\}$. By our hypothesis, we know

$$x_0 \preccurlyeq T^2x_0 = x_2.$$

Since T is a $\preccurlyeq$-nonincreasing selfmapping on $A \cup B$, we have

$$x_1 = Tx_0 \succcurlyeq Tx_2 = x_3,$$

$$x_2 = Tx_1 \preccurlyeq Tx_3 = x_4$$

and

$$x_3 = Tx_2 \succcurlyeq Tx_4 = x_5.$$

By induction, we have the following: for any $n \in \mathbb{N}$,

- $x_{2n-2} \preccurlyeq x_{2n}$,
- $x_{2n-1} \succcurlyeq x_{2n+1}$.

From our hypothesis again, since

$$x_2 = T^2x_0 \preccurlyeq Tx_0 = x_1,$$

we have

$$x_3 = Tx_2 \succcurlyeq Tx_1 = x_2,$$

$$x_4 = Tx_3 \preccurlyeq Tx_2 = x_3,$$

and

$$x_5 = Tx_4 \succcurlyeq Tx_3 = x_4.$$

So, by induction, we obtain the following: for any $n \in \mathbb{N}$,

- $x_{2n} \preccurlyeq x_{2n-1}$,
- $x_{2n+1} \succcurlyeq x_{2n}$.

Since $x_0 \in A$, $x_1 \in B$ and $x_0 \preccurlyeq x_1$, we get

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\leq d(x_0, x_1) - \varphi(d(x_0, x_1)) + \mu(\min\{d(Tx_0, x_1), d(x_0, Tx_1)\}) + \varphi(dist(A, B)) \\ &= d(x_0, x_1) - \varphi(d(x_0, x_1)) + \mu(\min\{d(x_1, x_1), d(x_0, x_2)\}) + \varphi(dist(A, B)) \\ &\leq d(x_0, x_1) - \varphi(d(x_0, x_1)) + \varphi(dist(A, B)). \end{aligned}$$

Since $x_2 \in A$, $x_1 \in B$ and $x_2 \preccurlyeq x_1$, we obtain

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq d(x_2, x_1) - \varphi(d(x_2, x_1)) + \mu(\min\{d(Tx_2, x_1), d(x_2, Tx_1)\}) + \varphi(dist(A, B)) \\ &= d(x_1, x_2) - \varphi(d(x_1, x_2)) + \mu(\min\{d(x_3, x_1), d(x_2, x_2)\}) + \varphi(dist(A, B)) \\ &\leq d(x_1, x_2) - \varphi(d(x_1, x_2)) + \varphi(dist(A, B)). \end{aligned}$$

By induction, we have

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})) + \varphi(dist(A, B)), \tag{2.2}$$

for all $n \in \mathbb{N}$. Clearly,

$$d(x_n, x_{n+1}) \geq dist(A, B) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Since φ is strictly increasing, by (2.3), we have

$$\varphi(d(x_n, x_{n+1})) \geq \varphi(dist(A, B)) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

For each $n \in \mathbb{N} \cup \{0\}$, by (2.2) and (2.4), we obtain

$$0 \leq \varphi(d(x_n, x_{n+1})) - \varphi(dist(A, B)) \leq d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})$$

which implies $\{d(x_n, x_{n+1})\}$ is nonincreasing. Hence

$$\gamma := \lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) \text{ exists.} \tag{2.5}$$

By (2.3) and (2.5), we know $\gamma \geq dist(A, B)$. We want to show

$$\gamma = dist(A, B).$$

Suppose $\gamma > dist(A, B)$. Because φ is strictly increasing, we get

$$\varphi(\gamma) > \varphi(dist(A, B)). \tag{2.6}$$

By (2.5), we have $\gamma \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. So

$$\varphi(\gamma) \leq \varphi(d(x_n, x_{n+1})) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

By (2.7) and (2.2), we have

$$\varphi(\gamma) \leq \varphi(d(x_n, x_{n+1})) \leq d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2}) + \varphi(dist(A, B))$$

for all $n \in \mathbb{N} \cup \{0\}$. The last inequality, (2.5) and (2.6) deduce

$$\varphi(\gamma) \leq \lim_{n \to \infty} \varphi(d(x_n, x_{n+1})) \leq \varphi(dist(A, B)) < \varphi(\gamma),$$

a contradiction. Hence it must be $\gamma = dist(A, B)$. Combining this with (2.5), we get

$$\lim_{n \to \infty} d(x_{n-1}, x_n) = \inf_{n \in \mathbb{N}} d(x_{n-1}, x_n) = dist(A, B). \tag{2.8}$$

Moreover, let us assume that $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$ and $\{x_{2n}\}_{n \in \mathbb{N} \cup \{0\}}$ has a convergent subsequence $\{x_{2n_k}\}$ in A. Then there exists $v \in A$ such that $x_{2n_k} \to v$ as $k \to \infty$. Since $Tv \in B$ and $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$, by (2.7), we have

$$dist(A, B) \leq d(x_{2n_k}, Tv) \leq d(x_{2n_k - 1}, v) \leq d(v, x_{2n_k}) + d(x_{2n_k}, x_{2n_k - 1}) \quad \text{for all } k \in \mathbb{N}.$$

By taking the limit from both sides in the last inequality as $k \to \infty$, we get $d(v, Tv) = dist(A, B)$. The proof is completed. □

The following existence theorems for best proximity points can be given immediately from Theorem 2.1.

Corollary 2.1. *Let $(X, d, \preccurlyeq)$ be a quasiordered metric space, A and B be nonempty subset of X and T be a cyclic $\preccurlyeq$-nonincreasing selfmapping on $A \cup B$. Given $\lambda \in \mathbb{R}$. Suppose that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \lambda \min\{d(Tx, y), d(x, Ty)\} + \varphi(dist(A, B))$$

for all $x \in A$ and $y \in B$ with $x \preccurlyeq y$, where $\varphi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function. Suppose that there exists* $x_0 \in A$ *such that* $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$. *Define an iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_{n+1} = Tx_n$ *for* $n \in \mathbb{N}\cup\{0\}$. *Then the following hold.*

(a) $d(x_{n+1}, x_{n+2}) \le d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})) + \varphi(dist(A, B))$ *for all* $n \in \mathbb{N} \cup \{0\}$*;*

(b) $\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.

Moreover, if $d(Tx, Ty) \le d(x, y)$ *for any* $x \in A$ *and* $y \in B$ *and* $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ *has a convergent subsequence in* A*, then there exists* $v \in A$ *such that* $d(v, Tv) = dist(A, B)$.

Proof. Define a function $\mu : [0,\infty) \to \mathbb{R}$ by $\mu(t) = \lambda t$ for all $t \ge 0$. Then the conclusions follows from Theorem 2.1 immediately. □

Corollary 2.2. *Let* $(X, d, \preccurlyeq)$ *be a quasiordered metric space,* A *and* B *be nonempty subset of* X *and* T *be a cyclic* $\preccurlyeq$*-nonincreasing selfmapping on* $A \cup B$*. Given a positive real number* k*. Suppose that*

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)) + (\min\{d(Tx, y), d(x, Ty)\})^k + \varphi(dist(A, B))$$

for all $x \in A$ and $y \in B$ with $x \preccurlyeq y$, where $\varphi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function. Suppose that there exists* $x_0 \in A$ *such that* $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$. *Define an iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_{n+1} = Tx_n$ *for* $n \in \mathbb{N}\cup\{0\}$. *Then the following hold.*

(a) $d(x_{n+1}, x_{n+2}) \le d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})) + \varphi(dist(A, B))$ *for all* $n \in \mathbb{N} \cup \{0\}$*;*

(b) $\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A, B)$.

Moreover, if $d(Tx, Ty) \le d(x, y)$ *for any* $x \in A$ *and* $y \in B$ *and* $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ *has a convergent subsequence in* A*, then there exists* $v \in A$ *such that* $d(v, Tv) = dist(A, B)$.

Proof. Define a function $\mu : [0,\infty) \to \mathbb{R}$ by $\mu(t) = t^k$ for all $t \ge 0$. Then the conclusions follows from Theorem 2.1 immediately. □

Clearly, [6, Theorem 2.2] is a special case for $\lambda = 0$ in Corollary 2.1.

Corollary 2.3. [6, Theorem 2.2] *Let* $(X, d, \preccurlyeq)$ *be an ordered metric space,* $A, B \in 2^X$ *and* T *a* $\preccurlyeq$*-nonincreasing selfmapping on* $A \cup B$ *such that* $T(A) \subseteq B$ *and* $T(B) \subseteq A$*. Suppose that there exists* $x_0 \in A$ *such that* $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$ *and*

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)) + \varphi(dist(A, B))$$

for all$x \in A$ *and* $y \in B$ *with* $x \preccurlyeq y$*, where* $\varphi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function. If* $x_{n+1} = Tx_n$ *and* $d_n = d(x_{n+1}, x_n)$ *for all* $n \ge 0$*, then* $d_n \to dist(A, B)$ *as* $n \to \infty$.

In Theorem 2.1, if we take $\varphi(t) = e^t$, then we have the following result.

Corollary 2.4. *Let* $(X, d, \preccurlyeq)$ *be a quasiordered metric space,* A *and* B *be nonempty subset of* X *and* T *be a cyclic* $\preccurlyeq$*-nonincreasing selfmapping on* $A \cup B$*. Suppose that*

$$d(Tx, Ty) \le d(x, y) - e^{d(x,y)} + \mu(\min\{d(Tx, y), d(x, Ty)\}) + e^{dist(A,B)}$$

for all $x \in A$ *and* $y \in B$ *with* $x \preccurlyeq y$, *where* $\mu : [0,\infty) \to \mathbb{R}$ *is a function satisfying* $\mu(0) \leq 0$. *Suppose that there exists* $x_0 \in A$ *such that* $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$. *Define an iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_{n+1} = Tx_n$ *for* $n \in \mathbb{N}\cup\{0\}$. *Then the following hold.*

(a) $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \varphi(d(x_n, x_{n+1})) + \varphi(dist(A,B))$ *for all* $n \in \mathbb{N}\cup\{0\}$;

(b) $\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = dist(A,B)$.

Moreover, if $d(Tx,Ty) \leq d(x,y)$ *for any* $x \in A$ *and* $y \in B$ *and* $\{x_{2n}\}_{n\in\mathbb{N}\cup\{0\}}$ *has a convergent subsequence in* A, *then there exists* $v \in A$ *such that* $d(v,Tv) = dist(A,B)$.

3. Some applications to the fixed point theory

Applying Theorem 2.1, we can establish the following new fixed point theorem on quasiordered metric spaces.

Theorem 3.1. *Let* $(X,d,\preccurlyeq)$ *be a quasiordered metric space and* $T : X \to X$ *be a* $\preccurlyeq$*-nonincreasing selfmapping on* X. *Suppose that*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) \tag{3.1}$$

for all $x,y \in X$ *with* $x \preccurlyeq y$, *where* $\varphi : [0,\infty) \to [0,\infty)$ *is a lower semicontinuous strictly increasing function satisfying* $\varphi(0) = 0$. *Suppose that there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$. *Define an iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *by* $x_{n+1} = Tx_n$ *for* $n \in \mathbb{N}\cup\{0\}$. *Then* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *is a Cauchy sequence in* X. *Moreover, if* (X,d) *is complete and one of the following conditions is satisfied:*

(H1) T *is continuous;*

(H2) $d(Tx,Ty) \leq d(x,y)$ *for all* $x,y \in X$;

(H3) T *is closed, that is* $\text{graph}(T) := \{(x,y) \in X \times X : y = Tx\}$ *is closed;*

(H4) *the map* $f : X \to [0,\infty)$ *defined by* $f(x) = d(x,Tx)$ *is lower semicontinuous,*

then the sequence $\{x_n\}_{n=0}^{\infty}$ *converges to a fixed point of* T.

Proof. Let $A = B = X$. Then $A \cup B = X$ and $dist(A,B) = 0$. Define $\mu : [0,\infty) \to \mathbb{R}$ by $\mu(t) = 0$ for all $t \geq 0$. So T is a cyclic $\preccurlyeq$-nonincreasing selfmapping on $A \cup B$ and (3.1) deduces

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) + \mu(\min\{d(Tx,y), d(x,Ty)\}) + \varphi(dist(A,B))$$

for all $x \in A$ and $y \in B$ with $x \preccurlyeq y$. Applying Theorem 2.1, we obtain

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n) = 0. \tag{3.2}$$

Now, we claim that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in X. Assume that our claim is not true. Then there exists $\delta > 0$ and sequences $\{m(k)\}$, $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k,$$

$$d(x_{n(k)}, x_{m(k)}) \geq \delta \tag{3.3}$$

and

$$d(x_{n(k)}, x_{m(k)-1}) < \delta. \tag{3.4}$$

From (3.2), there exists $N_0 \in \mathbb{N}$ such that $k > N_0$ implies

$$d(x_{n(k)}, x_{n(k)+1}) < \delta. \tag{3.5}$$

In the view of (3.3) and (3.5), we have $m(k) \neq n(k) + 1$ for all $k > N_0$. By taking into account (3.3) and (3.4), we get

$$\begin{aligned} 0 < \delta &\leq d(x_{n(k)}, x_{m(k)}) \\ &\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \delta + d(x_{m(k)-1}, x_{m(k)}), \end{aligned}$$

for $k > N_0$. Therefore, by (3.2) and the last inequality, we conclude that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \delta. \tag{3.6}$$

For each $k \in \mathbb{N}$, since

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}),$$

by taking into account (3.2), (3.6) and the last inequalities, we get

$$\limsup_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \delta.$$

Let $\lambda_k = d(x_{n(k)+1}, x_{m(k)+1})$, $k \in \mathbb{N}$. Then $\limsup_{k\to\infty} \lambda_k = \delta$. Hence there exists a subsequence $\{\lambda_{k(t)}\} \subset \{\lambda_k\}$ such that

$$\lim_{t\to\infty} \lambda_{k(t)} = \delta.$$

That is,

$$\lim_{t\to\infty} d(x_{n(k(t))+1}, x_{m(k(t))+1}) = \delta. \tag{3.7}$$

From (3.1), we have

$$d(x_{n(k(t))+1}, x_{m(k(t))+1}) \leq d(x_{n(k(t))}, x_{m(k(t))}) - \varphi(d(x_{n(k(t))}, x_{m(k(t))})) \quad \text{for all } t \in \mathbb{N}.$$

By (3.6), (3.7) and taking the limit superior from both sides of the last inequality as $t \to \infty$, we get

$$\lim_{t\to\infty} d(x_{n(k(t))+1}, x_{m(k(t))+1}) \leq \lim_{t\to\infty} d(x_{n(k(t))}, x_{m(k(t))}) - \liminf_{t\to\infty} \varphi(d(x_{n(k(t))}, x_{m(k(t))}))$$

and hence

$$\liminf_{t\to\infty} \varphi(d(x_{n(k(t))}, x_{m(k(t))})) = 0. \tag{3.8}$$

Since $d(x_{n(k(t))}, x_{m(k(t))}) \to \delta$ as $t \to \infty$ and φ is lower semicontinuous and strictly increasing, by (3.8), we get

$$0 = \varphi(0) < \varphi(\delta) \leq \liminf_{t\to\infty} \varphi(d(x_{n(k(t))}, x_{m(k(t))})) = 0,$$

a contradiction. Hence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ must be a Cauchy sequence in X.

Moreover, assume that (X,d) is complete. So there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. To finish the proof, we will show $v \in \mathcal{F}(T)$. Suppose that (H1) holds. Then

$$Tv = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = v.$$

If (H2) holds, then T is continuous and the desire follows from (H1) immediately. If (H3) holds, since T is closed, $Tx_n = x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$ and $x_n \to v$ as $n \to \infty$, we have

$$(v,v) = \lim_{n\to\infty}(x_n, x_{n+1}) \in \text{graph}(T),$$

and hence $Tv = v$. Suppose that (H4) holds. Since f is lower semicontinuous, by (3.3), we get

$$d(v,Tv) = f(v) \le \liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

which implies $d(v,Tv) = 0$ or $v \in \mathcal{F}(T)$. The proof is completed. □

Corollary 3.1. *Let $(X,d,\preccurlyeq)$ be a quasiordered metric space and $T : X \to X$ be a $\preccurlyeq$–nonincreasing selfmapping on X. Let λ* and *k be positive real numbers. Suppose that*

$$d(Tx,Ty) \le d(x,y) - \ln(1 + d(x,y))$$

for all $x,y \in X$ with $x \preccurlyeq y$. Suppose that there exists $x_0 \in X$ such that $x_0 \preccurlyeq T^2x_0 \preccurlyeq Tx_0$. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. Then $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence in X. Moreover, if (X,d) is complete and one of the following conditions is satisfied:

(H1) *T is continuous;*
(H2) *$d(Tx,Ty) \le d(x,y)$ for all $x,y \in X$;*
(H3) *T is closed, that is* graph$(T) := \{(x,y) \in X \times X : y = Tx\}$ *is closed;*
(H4) *the map $f : X \to [0,\infty)$ defined by $f(x) = d(x,Tx)$ is lower semicontinuous,*

then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of T.

Proof. Define a function $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \ln(1+t)$ for all $t \ge 0$. Then $\varphi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous strictly increasing function satisfying $\varphi(0) = 0$. Hence the conclusions follows from Theorem 3.1 immediately. □

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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