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τ-Generalization of Fixed Point Results for F-Contractions

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Abstract The aim of this paper is to extend wardowski fixed point results for F-contractions. In this paper we introduce the notion of α_*-η-τF-contraction for multivalued mappings and establish some new fixed point results for α_*-η-τF-contraction in a complete metric space. We extend the concept of F-contraction into α_*-η-τF-contraction and α_*-τ-F-contraction. Example are given to validate the results proved herein.

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1. Introduction

In 2012, Samet et al. [20] introduced a concept of $\alpha - \psi$- contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Thereafter, many papers have published on $\alpha - \psi-$ contractive mappings in various spaces. For more detail see [9–11, 14] and references therein.

In 2012, Wardowski [22] introduced a new type of contractions called F-contraction and proved new fixed point theorems concerning F -contraction. He generalized the Banach contraction principle in a different way than as it was done by different investigators. Afterwards Secelean [21] proved fixed point theorems consisting of F-contractions by Iterated function systems. Piri et al. [17] proved a fixed point result for F-Suzuki contractions for some weaker conditions on the self map of a complete metric space which generalizes the result of Wardowski.

Cosentino et al. [4] established some fixed point results of Hardy-Rogers-type for self-mappings on complete metric spaces or complete ordered metric spaces. Lately, Acar et al. [1] introduced the concept of generalized multivalued F-contraction mappings further Altun et al. [2] extended multivalued mappings with δ-Distance and established fixed point results in complete metric space. Sgroi et al.[18] established fixed point theorems

for multivalued F-contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued fixed point theorems including Nadler's. recently Ahmad et al. [3, 7] recalled the concept of F-contraction to obtain some fixed point, and common fixed point results in the context of complete metric spaces. Very recently Kutbi et al. [15] extend the concept of F-contraction to obtain some fixed point results in complete metric space.
Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set af all positive real numbers and by $\mathbb{N}$ the set of all positive integers.
recollect some essential notations, required definitions, and primary results coherent with the literature. For a nonempty set X, we denote by $N(X)$ the class of all nonempty subsets of X. Let (X,d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $D(x,A) = \inf\{d(x,y) : y \in A\}$. We denote by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $K(X)$ the class of all compact subsets of X, Let H be the Hausdorff metric induced by the metric d on X, that is

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B),\ \sup_{y\in B} D(y,A)\right\}$$

for every $A, B \in CB(X)$. If $T : X \longrightarrow CB(X)$ be a multi-valued. A point $q \in X$ is said to be a fixed point of T if $q \in Tq$.

Nadler [16] extended the Banach contraction principle to multivalued mappings.

Theorem 1 . [16] Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a multi-valued mapping such that for all $x, y \in X$

$$H(T(x),T(y)) \leq kd(x,y)$$

where $0 < k < 1$, Then T has a fixed point.

2. Preliminaries

In this section, we give some basic definitions, examples and fundamental results which play an essential role in proving our results.

Definition 2. [20] Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$.

Definition 3. [19] Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x,y) \geq \eta(x,y)$ implies that $\alpha(Tx,Ty) \geq \eta(Tx,Ty)$.

If $\eta(x,y) = 1$, then above definition reduces to definition 2. If $\alpha(x,y) = 1$, then T is called an η-subadmissible mapping.

Definition 4. [8] Let (X,d) be a metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is $\alpha - \eta$-continuous mapping on (X,d) if for given $x \in X$, and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty,\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

Definition 5. [6] Let (X,d) be ametric space, $T : X \to 2^X$ be a given closed-valued multifunction and $\alpha : X \times X \longrightarrow [0,+\infty)$. We say that T is called α_*-admissible whenever $\alpha(x,y) \geq 1$ implies that $\alpha_*(Tx,Ty) \geq 1$.

Hussain et al. [9] modified the notions of α_*-admissible and α_*-ψ-contractive mappings as follows:

Definition 6. [9] Let $T : X \to 2^X$ be a multifunction, $\alpha,\eta : X \times X \to [0,+\infty)$ be two functions where η is bounded. We say that T is α_*-admissible mapping with respect to η if $\alpha(x,y) \geq \eta(x,y)$ implies $\alpha_*(Tx,Ty) \geq \eta_*(Tx,Ty)$, $x,y \in X$, where $\alpha_*(A,B) = \inf\{\alpha(x,y) : x \in A,\ y \in B\}$ and $\eta_*(A,B) = \sup\{\eta(x,y) : x \in A,\ y \in B\}$.

If $\eta(x,y) = 1$ for all $x,y \in X$, then this definition reduces to Definition 4.1[9] . In the case $\alpha(x,y) = 1$ for all $x,y \in X$, T is called η_*-subadmissible mapping.

Definition 7. [12] Let (X,d) be a metric space. Let $T : X \to CL(X)$ and $\alpha : X \times X \to [0,+\infty)$ be two functions. We say that T is α-continuous multivalued mapping on $(CL(X),H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $\lim_{n\longrightarrow\infty} d(x_n,x) = 0$, $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \Longrightarrow \lim_{n\longrightarrow\infty} H(Tx_n,Tx) = 0$.

In 1962, Edelstein proved the following version of the Banach contraction principle.

Theorem 8. [5]. Let (X,d) be a metric space and $T : X \to X$ be a self mapping. Assume that

$$d(Tx,Ty) < d(x,y), \text{ holds for all } x,y \in X \text{ with } x \neq y.$$

Then T has a unique fixed point in X.

Klim et al. [13] defined the F-contraction as follows.

Definition 9. [13] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F contraction if there exists $\tau > 0$ such that

$$\forall x,y \in X,\ d(Tx,Ty) > 0 \Rightarrow \tau + F\left(d(Tx,Ty)\right) \leq F\left(d(x,y)\right), \tag{1.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:
(F1) F is strictly increasing, i.e. for all $x,y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) There exists $k \in (0,1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

We denote by Δ_F, the set of all functions satisfying the conditions (F1)-(F3).

Example 10. [22] Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln\alpha$. It is clear that F satisfied (F1)-(F2)-(F3) for any $k \in (0,1)$. Each mapping $T : X \to X$ satisfiying (1.1) is an F-contraction such that

$$d(Tx,Ty) \leq e^{-\tau} d(x,y), \text{ for all } x,y \in X,\ Tx \neq Ty.$$

It is clear that for $x,y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \leq e^{-\tau} d(x,y)$, also holds, i.e. T is a Banach contraction.

Example 11. [22] If $F(\alpha) = \ln\alpha + \alpha$, $\alpha > 0$ then F satisfies (F1)-(F3) and the condition (1.1) is of the form

$$\frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \text{ for all } x,y \in X,\ Tx \neq Ty.$$

Remarks 12. From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [22] stated a modified version of the Banach contraction principle as follows.

Theorem 13 . [22] Let (X,d) be a complete metric space and let $T : X \to X$ be an F contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.Lately, Acar et al. [1] introduced the concept of generalized multivalued F-contraction mappings and established a fixed point result, which was a proper generalization of some multivalued fixed point theorems including Nadler's.

Definition 14. [1] Let (X,d) be a metric space and $T : X \longrightarrow CB(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that

$$x, y \in X,\ H(Tx,Ty) > 0 \Longrightarrow \tau + F(H(Tx,Ty)) \le F(M(x,y)),$$

where

$$M(x,y) = \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{1}{2}[D(x,Ty) + D(y,Tx)]\}.$$

Theorem 15. [1] Let (X,d) be a complete metric space and $T : X \longrightarrow K(X)$ be a generalized multivalued F-contraction. If T or F is continuous, then T has a fixed point in X.

We now introduce the concept of α-η-continuous for multivalued mappings in metric spaces.

Definition 16. Let (X,d) be a metric space. Let $T : X \to CB(X)$ and $\alpha : X \times X \to [0,+\infty)$ be function. We say that T is α_*-admissible multivalued mapping on $(CB(X),H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $x_n \overset{d}{\longrightarrow} x$ as $n \to \infty$, $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \Rightarrow Tx_n \overset{H}{\to} Tx$, that is $\lim\limits_{n\longrightarrow\infty} d(x_n,x) = 0$ and $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N} \Longrightarrow \lim\limits_{n\longrightarrow\infty} H(Tx_n,Tx) = 0$.

Definition 17. Let (X,d) be a metric space. Let $T : X \to CB(X)$ and $\alpha, \eta : X \times X \to [0,+\infty)$ be two functions. We say that T is $\alpha - \eta$-continuous multivalued mapping on $(CB(X),H)$ if for given $x \in X$, and sequence $\{x_n\}$ with $x_n \overset{d}{\longrightarrow} x$ as $n \to \infty$, $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N} \Rightarrow Tx_n \overset{H}{\to} Tx$, that is $\lim\limits_{\longrightarrow\infty} d(x_n,x) = 0$ and $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N} \Longrightarrow \lim\limits_{n\longrightarrow\infty} H(Tx_n,Tx) = 0$.

3. Fixed Point Results for α_*-τ-F-Contraction

In this section, we define a contraction called α_*-τ-F-contraction for multivalued mapping and obtain some new fixed point theorems for such contraction in the setting of complete metric spaces. We define multivalued α_*-τ-F-contraction as follows:

Definition 18. Let (X,d) be a metric space and $T : X \longrightarrow CB(X)$ be an α_*-admissible multivalued mapping. Also suppose that $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing function. We say that T is multivalued α_*-τ-F-contraction if for $x, y \in X$, and $H(Tx,Ty) > 0$ we have

$$2\tau(M(x,y)) + \alpha_*(Tx,Ty)F(H(Tx,Ty)) \le F(M(x,y)), \tag{3.1}$$

where

$$M(x,y) = \max\{d(x,y), D(x,Tx), D(y,Ty)\}$$

and $F \in \Delta_F$.

Our main result is the following.

Theorem 19. Let (X,d) be a complete metric space. Let $T : X \longrightarrow CB(X)$ satisfying the following assertions:
(i) T is an α_*-admissible multivalued mapping;
(ii) T is multivalued α_*-τ-F-contraction;
(iii) there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$;
(iv) $\forall_{t\geq 0} \liminf_{s\to t^+} \tau(s) > 0$;
(v) T is continuous.
Then T has a fixed point in X.

Proof. Let $x_0 \in X$, such that $\alpha_*(x_0, Tx_0) \geq 1$. Since T is an α_*-admissible mapping then there exists $x_1 \in Tx_0$ such that

$$\alpha_*(x_0, Tx_0) \geq 1. \tag{3.2}$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T. So, we assume that $x_0 \neq x_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F(hH(Tx_0,Tx_1)) < F(H(Tx_0,Tx_1)) + \tau(\max\{d(x_0,x_1)D(x_0,Tx_0), D(x_1,Tx_1)\})$$

Now from $D(x_1,Tx_1) < hH(Tx_0,Tx_1)$, we deduce that there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) \leq hH(Tx_0,Tx_1)$. Consequently, we obtain

$$\begin{aligned} F(D(x_1,Tx_1)) &\leq F(hH(Tx_0,Tx_1)) \\ &< F(H(Tx_0,Tx_1)) + \tau(\max\{d(x_0,x_1)D(x_0,Tx_0), D(x_1,Tx_1)\}). \end{aligned}$$

Which implies

$$\begin{aligned} & 2\tau(\max\{d(x_0,x_1)D(x_0,Tx_0), D(x_1,Tx_1)\}) + F(d(x_1,x_2)) \\ \leq\ & 2\tau(\max\{d(x_0,x_1)D(x_0,Tx_0), D(x_1,Tx_1)\}) + F(H(Tx_0,Tx_1)) + \\ & \tau(\max\{D(x_0,Tx_0), D(x_1,Tx_1)\}) \\ \leq\ & F(\max\{D(x_0,Tx_0), D(x_1,Tx_1)\}) + \tau(\max\{D(x_0,Tx_0), D(x_1,Tx_1)\}). \end{aligned}$$

In this case $\max\{D(x_0,Tx_0), D(x_1,Tx_1)\} = D(x_1,Tx_1)$ is impossible, because

$$\begin{aligned} F(D(x_1,Tx_1)) &\leq \alpha_*(Tx_0,Tx_1)F(H(Tx_0,Tx_1)) \\ &\leq F(D(x_1,Tx_1)) - \tau(D(x_1,Tx_1)) \\ &< F(D(x_1,Tx_1)). \end{aligned}$$

Which is a contradiction. Thus

$$\begin{aligned} F(d(x_1,Tx_1)) &\leq \alpha_*(Tx_0,Tx_1)F(H(Tx_0,Tx_1)) \\ &\leq F(D(x_0,Tx_0)) - \tau(D(x_0,Tx_0)). \end{aligned}$$

By continuing this process, we obtain a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$,

$$\alpha_*(x_{n-1},Tx_{n-1}) = \alpha_*(x_{n-1},x_n) \geq 1,$$

Therefore, we obtain

$$\begin{aligned} F\left(d(x_n, Tx_n)\right) &\leq \alpha_*(Tx_{n-1}, Tx_n)F\left(d(Tx_{n-1}, Tx_n)\right) \\ &\leq F\left(d(x_{n-1}, x_n)\right) - \tau(d(x_{n-1}, x_n)). \end{aligned}$$

So, sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence and hence convergent. Then we prove that $d(x_n, x_{n+1}) \to 0$.From (iii) there exists $c > 0$ and $n \in \mathbb{N}$ such that $\tau\left(d(x_n, x_{n+1})\right) > c$ for all $n > n_0$. Thus, we obtain

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n)) \\ &\leq F(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-1}, x_n)) \\ &\;\;\vdots \\ &\leq F(d(x_0, x_1)) - \tau(d(x_0, x_1)) - \cdots - \tau(d(x_{n-1}, x_n)) \\ &= F(d(x_0, x_1)) - (\tau(d(x_0, x_1)) + \cdots + \tau(d(x_{n_0-1}, x_{n_0}))) \\ &\quad - (\tau(d(x_{n_0}, x_{n_0+1})) + \cdots + \tau(d(x_{n-1}, x_n))) \\ &\leq F(d(x_0, x_1)) - (n - n_0)\, c3.3 \end{aligned} \tag{3.1}$$

Since $F \in \Delta_F$, so by taking limit as $n \longrightarrow \infty$in (3.3), we deduce

$$\lim_{n \longrightarrow \infty} F(d\left(x_n, x_{n+1}\right)) = -\infty \iff \lim_{n \longrightarrow \infty} d\left(x_n, x_{n+1}\right) = 0. \tag{3.4}$$

Now from (F_3), there exists $0 < k < 1$ such that

$$\lim_{n \longrightarrow \infty} \left[d\left(x_n, x_{n+1}\right)\right]^k F(d\left(x_n, x_{n+1}\right)) = 0. \tag{3.5}$$

By (3.3), we have

$$\begin{aligned} & d\left(x_n, x_{n+1}\right)^k F(d\left(x_n, x_{n+1}\right)) - d\left(x_n, x_{n+1}\right)^k F\left(d\left(x_0, x_1\right)\right) 3.6 \qquad (3.2) \\ \leq\; & d\left(x_n, x_{n+1}\right)^k \left[F\left(d\left(x_0, x_1\right) - \left(n - n_0\right) c\right)\right] - d\left(x_n, x_{n+1}\right)^k F\left(d\left(x_0, x_1\right)\right) \\ =\; & -\left(n - n_0\right) c \left[d\left(x_n, x_{n+1}\right)\right]^k \leq 0. \end{aligned}$$

Letting $n \longrightarrow \infty$ in (3.6) and applying (3.4) and (3.5), we have,

$$\lim_{n \longrightarrow \infty} n \left[d\left(x_n, x_{n+1}\right)\right]^k = 0. \tag{3.7}$$

We observe that from (3.7), then there exists $n_1 \in \mathbb{N}$, such that $n\left(d(x_n, x_{n+1})\right)^k \leq 1$ for all $n \geq n_1$, we get

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1. \tag{3.8}$$

Now, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then, by the triangle inequality and from (3.8) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{m-1}, x_m) \quad (3.9) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent. By taking limit as $n \to \infty$, in (3.9), we have

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x^* \in X$ such that $\lim_{n\longrightarrow\infty} d(x_n, x^*) = 0$. By (v) T is continuous, we get

$$\lim_{n\longrightarrow\infty} H(Tx_n, Tx^*) = 0.$$

Now we obtain

$$D(x^*, Tx^*) = \lim_{n\longrightarrow\infty} D(x_{n+1}, Tx^*) \leq \lim_{n\longrightarrow\infty} H(Tx_n, Tx^*) = 0.$$

Therefore, $x^* \in Tx^*$ and hence T has a fixed point.

Theorem 20 . Let (X, d) be a complete metric space. Let $T : X \longrightarrow CB(X)$ satisfying the following assertions:

(i) T is multivalued α_*-admissible mapping;

(ii) T is multivalued α_*-τ-F-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$;

(iv) $\forall_{t\geq 0} \liminf_{s\to t^+} \tau(s) > 0$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha_*(x_n, x_{n+1}) \geq 1$ with $x_n \to x^*$ as $n \to \infty$ then $\alpha_*(x_n, x^*) \geq 1$ holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof. As similar lines of the Theorem 19, Since, by (v), $\alpha_*(x_{n+1}, x^*) \geq 1$ for all $n \in \mathbb{N}$. Then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(x_{n_k+1}, x^*) \geq 1. \quad (3.10)$$

From (2.1), we have

$$\tau(M(x_{n_k}, x^*)) + \alpha(Tx_{n_k}, Tx^*)F(H(Tx_{n_k}, Tx^*)) \leq F(M(x_{n_k}, x^*))$$

This implies

$$\begin{aligned} &\tau(\max\{d(x_{n_k}, x^*), D(x_{n_k}, Tx_{n_k}), D(x^*, Tx^*)\}) + \alpha(Tx_{n_k}, Tx^*)F(H(Tx_{n_k}, Tx^*)) \\ \leq\ &F(\max\{d(x_{n_k}, x^*), D(x_{n_k}, Tx_{n_k}), D(x^*, Tx^*)\}) \end{aligned}$$

Using the continuity of F and the fact that

$$\lim_{k\to\infty} d(x_{n_k}, x^*) = 0 = \lim_{k\to\infty} d(x_{n_k+1}, x^*) \quad (3.11)$$

we obtain

$$\tau\left(D(x^*, Tx^*)\right) + F\left(D(x^*, Tx^*)\right) \leq F\left(D(x^*, Tx^*)\right).$$

Which is a contradiction. Therefore, $x^* \in Tx^*$, implies x^* is a fixed point of T.

In the following we extend the Wardowski type fixed point theorem.

4. Fixed Point Results for α_*-η-τF-Contraction

In this section, we extend α_*-τ-F-contraction into α_*-η-τF-contraction and obtained some new fixed point theorems in the setting of complete metric space. We define α_*-η-τF-contraction as follows:

Definition 21. Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$ be an α_*-admissible multivalued mapping with respect to η_*. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty), \tau : \mathbb{R}_+ \to \mathbb{R}_+$ be three functions. We say that T is multivalued α_*-η-τF-contraction if for all $x, y \in X$, with $\eta_*(x, Tx) \leq \alpha_*(x, y)$ and $H(Tx, Ty) > 0$, we have

$$2\tau(M(x, y)) + F\left(H(Tx, Ty)\right) \leq F\left(M(x, y)\right) \tag{3.12}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Tx), D(y, Ty)\right\}$$

and $F \in \Delta_F$.

Now we state our result.

Theorem 22. Let (X, d) be a complete metric space. Let $T : X \longrightarrow CB(X)$ satisfying the following assertions:

(i) T is multivalued α_*-admissible mapping with respect to η;
(ii) T is multivalued α_*-η-τF-contraction;
(iii) there exists $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$;
(iv) $\forall_{t \geq 0} \liminf_{s \to t^+} \tau(s) > 0$;
(v) T is $\alpha - \eta$-continuous multivalued mapping.

Then T has a fixed point in X.

Proof. Let $x_0 \in X$, such that $\alpha_*(x_0, Tx_0) \geq \eta_*(x_0, Tx_0)$. Since T is an α_*-admissible mapping with respect to η then there exists $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha_*(x_0, Tx_0) \geq \eta_*(x_0, Tx_0) = \eta(x_0, x_1). \tag{3.13}$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T. So, we assume that $x_0 \neq x_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F\left(hH\left(Tx_0, Tx_1\right)\right) < F\left(H\left(Tx_0, Tx_1\right)\right) + \tau\left(\max\left\{d(x_0, x_1) D(x_0, Tx_0), D(x_1, Tx_1)\right\}\right)$$

Now from $D\left(x_1, Tx_1\right) < hH\left(Tx_0, Tx_1\right)$, we deduce that there exists $x_2 \in Tx_1$ such that $d\left(x_1, x_2\right) \leq hH\left(Tx_0, Tx_1\right)$. Consequently, we obtain

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(hH(Tx_0, Tx_1)) \\ &< F(H(Tx_0, Tx_1)) + \tau\left(\max\left\{d(x_0, x_1) D(x_0, Tx_0), D(x_1, Tx_1)\right\}\right). \end{aligned}$$

Which implies

$$\begin{aligned} & 2\tau\left(\max\left\{d(x_0,x_1)D(x_0,Tx_0),D(x_1,Tx_1)\right\}\right)+F(d(x_1,x_2)) \\ \leq\; & 2\tau\left(\max\left\{d(x_0,x_1)D(x_0,Tx_0),D(x_1,Tx_1)\right\}\right)+F\left(H(Tx_0,Tx_1)\right)+ \\ & \tau\left(\max\left\{D(x_0,Tx_0),D(x_1,Tx_1)\right\}\right) \\ \leq\; & F\left(\max\left\{D(x_0,Tx_0),D(x_1,Tx_1)\right\}\right)+\tau\left(\max\left\{D(x_0,Tx_0),D(x_1,Tx_1)\right\}\right). \end{aligned}$$

In this case $\max\{D(x_0,Tx_0),D(x_1,Tx_1)\}=D(x_1,Tx_1)$ is impossible, because

$$\begin{aligned} F\left(D(x_1,Tx_1)\right) &\leq \alpha_*(Tx_0,Tx_1)F\left(H(Tx_0,Tx_1)\right) \\ &\leq F\left(D(x_1,Tx_1)\right)-\tau(D(x_1,Tx_1)) \\ &< F\left(D(x_1,Tx_1)\right). \end{aligned}$$

Which is a contradiction. Thus

$$\begin{aligned} F\left(D(x_1,Tx_1)\right) &\leq \alpha_*(Tx_0,Tx_1)F\left(H(Tx_0,Tx_1)\right) \\ &\leq F\left(D(x_0,Tx_0)\right)-\tau(D(x_0,Tx_0)). \end{aligned}$$

By continuing this process, we obtain a sequence $\{x_n\}\subset X$ such that $x_n\notin Tx_n$, $x_{n+1}\in Tx_n$,

$$\eta(x_{n-1},x_n)=\eta_*(x_{n-1},Tx_{n-1})\leq\alpha_*(x_{n-1},Tx_{n-1})=\alpha(x_{n-1},x_n). \tag{3.14}$$

rest of proof follows similar lines as in Theorem 19.

Corollary 23. [8] Let (X,d) be a complete metric space. Let $T:X\to X$ be a self-mapping satisfying the following assertions:
(i) T is an α-admissible mapping with respect to η;
(ii) If for $x,y\in X$ with $\eta(x,Tx)\leq\alpha(x,y)$ and $d(Tx,Ty)>0$, we have

$$\tau+F\left(d(Tx,Ty)\right)\leq F\left(d(x,y)\right).$$

where $\tau>0$ and $F\in\Delta_F$.
(iii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$;
(iv) T is an $\alpha-\eta$-continuous.
Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y)\geq\eta(x,y)$ for all $x,y\in Fix(T)$.

Example 24. Let $X=[0,1]$, and $T:X\to CB(X)$ be defined as $Tx=\left[0,\frac{x}{3}\right]$ and d be the usual metric on X. Define $\alpha,\eta:X\times X\longrightarrow[0,\infty)$, $\tau:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ and $F:\mathbb{R}_+\longrightarrow\mathbb{R}$ by $\alpha(x,y)=\frac{1}{2}$, $\eta(x,y)=\frac{1}{4}$, $\tau(t)=\ln\left(\sqrt{t}\right)$ and $F(t)=\ln(t)+t$ for all $t>0$. Then for all $x,y\in X$, $Tx\neq Ty$, we obtain

$$\begin{aligned} & \tau\left(M(x,y)\right)+F(d(Tx,Ty)) \\ =\; & \frac{1}{2}\ln(t)+\ln(d(Tx,Ty))+d(Tx,Ty) \\ \leq\; & \ln(t)+\ln(\frac{1}{3}|y-x|)+\frac{1}{3}|y-x| \\ \leq\; & \ln(t)+\ln(\frac{1}{t})+\ln(\frac{1}{3}|y-x|)+\frac{1}{3}|y-x| \\ =\; & F(d(x,y)) \\ \leq\; & F(M(x,y)). \end{aligned}$$

Therefore T is an α_*-η-τF-contraction. Thus all conditions of above theorems are satisfied and 0 is a fixed point of T.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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