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Some Fixed Point Theorems and Common Fixed Point Theorems in Metric Space Involving a Graph

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Abstract In this paper, we introduce the notion of (A_φ, G)-contractions to generalize and extend the notion of A-contractions. We investigate the existence and uniqueness of fixed point for such contractions in metric space involving a graph. Also, under suitable assumptions, we prove some common fixed point theorems for two self-maps in metric space with a directed graph. Our results extend, improve, and generalize some recent results in the literature, including the results of Jachmyski [Proc.Amer.Math.Soc., 136(2008), 1359-1373.] and Akram et al. [Novi.Sad.J.Math., 38(1),(2008),25-33]. Moreover, we present some examples to validate and illustrate our results.

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Introduction

It is well-known that the contraction mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922 in [1], is one of the the most important theorems in classical functional analysis. This contraction mapping principle has been generalized in many directions. Recently, a very interesting generalization was obtained by Jachymski [2]. In fact, Jachymski [2] defined a concept of fixed point theory in some general structures by using the context of metric spaces endowed with a graph. After then Abbas [3] *et al.* obtained common fixed point results for such maps without appealing to any form of commutativity conditions defined on a partial metric space endowed with a directed graph. Beg [4] attained sufficient conditions for the existence of a common fixed point of the set-valued mappings in metric space involving a graph. Also, various authors proved fixed point theorems in abstract spaces endowed with a graph [5-15]. On the otherhand, Akram [16] introduced a new class of contraction maps, called $A-$contraction, which is a proper superclass of Kannan's [17], Bianchini's [18] and Reich's [19] type contractions.

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In this paper, we introduce the notion of (A_φ, G)-contractions to generalize and extend the notion of A-contractions. We investigate the existence and uniqueness of fixed point for such contractions in metric space with a graph. Also, under suitable assumptions, we prove some common fixed point theorems for two self-maps in metric space involving a directed graph. Our results extend, improve, and generalize some recent results in the literature.

1. Preliminaries

In this section, we list some fundamental definitions that are useful tool in consequent analysis.

Definition 1. [16] Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and A be the set of all functions $\alpha : \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ satisfying

i. α is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}_+^3$).
ii. $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b.

Now we give the class of contraction which is called A-contraction:

Definition 2. [16] A self-map T on a metric space X is said to be A-contraction if it satisfies the condition:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

Δ denote the diagonal of the Cartesian product $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. Assume that G has no parallel edges, so one can identify G with the pair $(V(G), E(G))$.

The conversion of a graph G is denoted by G^{-1} and which is a graph obtained from G by reversing the direction of edges. Hence

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

By $\tilde{G}$, we denote the undirected graph obtained from G by omitting the direction of edges. Indeed; it is more convenient to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

For any $x, y \in V', (x, y) \in E'$ such that $V' \subseteq V(G), E' \subseteq E(G)$, then (V', E') is called a subgraph of G. If x and y are vertices in a graph G, then a path from x to y of length $N (N \in \mathbb{N})$ is a sequence $(x_i)_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected. Some basic notations related to connectivity of graphs can be found in [20].

If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G) = [x]_G$ where $[x]_G$ denotes the equivalence class of relation $\Re$ defined on $V(G)$ by the rule

$$y \Re z \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

If $T : X \to X$ is an operator, then we denote

$$X_T = \{x \in X : (x, Tx) \in E(G)\}.$$

Definition 3. [2] A mapping $T : X \to X$ is a Banach $G-$contraction or simply $G-$contraction if the following conditions hold;

i. T preserves edges of G;

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$$

for all $x, y \in X$,

ii. T decreases weights of edges of G if there exists an $\eta \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow d(Tx, Ty) <\leq \eta d(x, y)$$

for all $x, y \in X$.

Definition 4. A mapping $T : X \to X$ is called orbitally G-continuous if, for all $x, y \in X$ and any sequence $(n_p)_{p\in\mathbf{N}}$ of positive integers,

$$T^{n_p}x \to y, \ \left(T^{n_p}x, T^{n_p+1}x\right) \in E(G) \text{ imply } T(T^{n_p}x) \to Ty \text{ as } p \to \infty.$$

Definition 5. [15] The graph G is called a (C)-graph whenever for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 0$.

Now we recall the following definitions which use them in the sequel.

Definition 6. [21] Let $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$ be a class of function, which satisfies the following conditions.

i. $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;

ii. $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 for all $t > 0$;

iii. $\sum\limits_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$;

If conditions (i-ii) hold then φ is called a comparison function, and, the comparison function satisfies (iii), then φ is called a strong comparison function.

Remark 1. [21] Any strong comparison function is a comparison function.

Remark 2. [21] If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0.

2. Fixed Point Theorems for (A_φ, G)-Contraction in Metric Space with a Graph

Initially, we establish some fixed-point theorems in metric space with a graph by defining (A_φ, G)-contraction.

Definition 7. Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and A_φ be the set of all functions $\alpha : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying

i. α is continuous on the set $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3_+$).

ii. for all $u, v \in \mathbb{R}_+$, $u \leq \alpha(u, v, v)$ or $u \leq \alpha(v, u, v)$ or $u \leq \alpha(v, v, u)$, then $u \leq \varphi(v)$, where φ is a strong comparison function.

In this definition, if we take $\varphi(t) = kt$ as $k \in (0, 1)$ for all $t > 0$, then we obtain $\alpha \in A$.

Definition 8. We say that a mapping $T : X \to X$ is a (A_φ, G)-contraction if it satisfy the following conditions;

i. T preserves edges of G, i.e. $((x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G))$, $\forall x, y \in X$;
ii. there exist some $\alpha \in A_\varphi$ such that

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), (y, Ty))$$

for each $(x, y) \in E(G)$.

Remark 3. Let X be a metric space endowed with a graph G and $T : X \to X$ be a (A_φ, G)-contraction for which there exists $x_0 \in X$ such that $Tx_0 \in [x_0]_{\tilde{G}}$, then

i. T is both a (A_φ, G^{-1})-contraction and a $\left(A_\varphi, \tilde{G}\right)$-contraction,
ii. $[x_0]_{\tilde{G}}$ is $T - invariant$ and $T\left|_{[x_0]_{\tilde{G}}}\right.$ is a $\left(A_\varphi, \tilde{G}_{x_0}\right)$-contraction.

Lemma 1. Let X be a metric space endowed with a graph G and $T : X \to X$ be a (A_φ, G)-contraction. Then given $x_0 \in X_T$, there exists $r(x_0, Tx_0) \geq 0$ such that

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq \varphi^n\left(r(x_0, Tx_0)\right)$$

for all $n \in \mathbb{N}$, where $r(x_0, Tx_0) = d(x_0, Tx_0)$.

*Proof.*Let $x_0 \in X_T$, *i.e.*, $(x_0, Tx_0) \in E(G)$. Then an easy induction shows that $\left(T^n x_0, T^{n+1} x_0\right) \in E(G)$ for all $n \in \mathbb{N}$. Thus, using definition of α,we get

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq \alpha\left(d\left(T^{n-1} x_0, T^n x_0\right), d\left(T^{n-1} x_0, T^n x_0\right), d\left(T^n x_0, T^{n+1} x_0\right)\right)$$

$$\leq \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right)$$

In this way for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d\left(T^n x_0, T^{n+1} x_0\right) &\leq \varphi\left(d\left(T^{n-1} x_0, T^n x_0\right)\right) \\ &\leq \varphi\left(\varphi\left(d\left(T^{n-2} x_0, T^{n-1} x_0\right)\right)\right) \\ &= \varphi^2\left(d\left(T^{n-2} x_0, T^{n-1} x_0\right)\right) \\ &\vdots \\ &\leq \varphi^n\left(d(x_0, Tx_0)\right) = \varphi^n\left(r(x_0, Tx_0)\right). \end{aligned}$$

Thus,

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq \varphi^n\left(r(x_0, Tx_0)\right)$$

for all $n \in \mathbb{N}$, where $r(x_0, Tx_0) = d(x_0, Tx_0)$.

Theorem 1. Let (X, d) be a complete metric space endowed with a graph and T be self-map on X. Suppose that:

i. G is weakly connected and (C)-graph;
ii. T is a $\left(A_\varphi, \tilde{G}\right)$-contraction;

ii. X_T is nonempty.

Then T is a PO.

Proof. Let $x_0 \in X_T$, then $\left(T^n x_0, T^{n+1} x_0\right) \in E(G)$ for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ with $m > n$. As from lemma 1, we get

$$d\left(T^n x_0, T^{n+1} x_0\right) \leq \varphi^n\left(r\left(x_0, T x_0\right)\right).$$

Since from definition 6 (ii), we have $\lim_{n \to \infty} \varphi^n\left(d\left(x_0, T x_0\right)\right) = 0$. For a given $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \geq n_0$.

$$\varphi^n\left(d\left(x_0, T x_0\right)\right) < \varepsilon - \varphi(\varepsilon).$$

Hence

$$d\left(T^n x_0, T^{n+1} x_0\right) < \varepsilon - \varphi(\varepsilon), \tag{2.1}$$

for all $n \geq n_0$.

Now, for any $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we claim that

$$d\left(T^n x_0, T^m x_0\right) < \varepsilon. \tag{2.2}$$

We prove the inequality (2.2) by induction on m. The inequality (2.2) holds for $m = n+1$, by using (2.1). Assume that (2.2) holds for $m = k$. So that $m = k+1$, we have

$$\begin{aligned} d\left(T^n x_0, T^m x_0\right) &\leq d\left(T^n x_0, T^{n+1} x_0\right) + d\left(T^{n+1} x_0, T^{k+1} x_0\right) \\ &< \varepsilon - \varphi(\varepsilon) + d\left(T^{n+1} x_0, T^{k+1} x_0\right) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi\left(d\left(T^n x_0, T^k x_0\right)\right) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\ &= \varepsilon \end{aligned}$$

By induction on m, we conclude that (2.2) holds for $m > n \geq n_0$. Thus, $\{T^n x_0\}$ is Cauchy sequence, by the completeness of X, $\{T^n x_0\}$ is converges to $x^* \in X$.

Next we prove that x^* is a fixed point of T. $T^n x_0 \to x^*$ and $\left(T^n x_0, T^{n+1} x_0\right) \in E(G)$ for all $n \in \mathbb{N}$. Since G is a (C)-graph, there exists a subsequence $\{T^{n_p} x_0\}$ of $\{T^n x_0\}$ such that $\left(T^{n_p} x_0, x^*\right) \in E(G)$ for each $p \in \mathbb{N}$. Also, $\left(T^{n_p} x_0, x^*\right) \in E\left(\tilde{G}\right)$ for each $p \in \mathbb{N}$. Hence,

$$d\left(T^{n_p+1} x_0, T x^*\right) \leq \alpha\left(d\left(T^{n_p} x_0, x^*\right), d\left(T^{n_p} x_0, T^{n_p+1} x_0\right), d\left(x^*, T x^*\right)\right)$$

taking limit as $p \to \infty$, we get

$$d\left(x^*, T x^*\right) \leq \alpha\left(0, 0, d\left(x^*, T x^*\right)\right).$$

From definition of α, $d\left(x^*, T x^*\right) \leq \varphi(0) = 0$. Thus, $T x^* = x^*$.

Finally, we prove that x^* is a unique fixed point. Suppose that $T^n x_0 \to y^*$ and $y^* = T y^*$ with $y^* \neq x^*$. $T^n x_0 \to x^*$, $T^n x_0 \to y^*$ and $\left(T^n x, T^{n+1} x\right) \in E(G)$ for all $n \in \mathbb{N}$. Also G is a (C)-graph, then there exists a subsequence $\{T^{n_p} x_0\}$ of $\{T^n x_0\}$ such that $\left(T^{n_p} x_0, x^*\right) \in E(G)$ and $\left(T^{n_p} x_0, y^*\right) \in E(G)$ for each $p \in \mathbb{N}$. Furthermore,

$(T^{n_p}x_0, x^*) \in E\left(\tilde{G}\right)$ and $(T^{n_p}x_0, y^*) \in E\left(\tilde{G}\right)$ for each $p \in \mathbb{N}$ because of the fact that G is weakly connected $(x^*, y^*) \in E\left(\tilde{G}\right)$.

$$d\left(x^*, y^*\right) = d\left(Tx^*, Ty^*\right) \leq \alpha\left(d\left(x^*, y^*\right), d\left(x^*, Tx^*\right), d\left(y^*, Ty^*\right)\right)$$

$$\leq \alpha\left(d\left(x^*, y^*\right), 0, 0\right).$$

by definition of α, $d\left(x^*, y^*\right) \leq \varphi\left(0\right) = 0$. Thus, $y^* = x^*$.

In this theorem, by replacing the property (C)-graph with T is orbitally G-continuous, we have following theorem.

Theorem 2. Let (X, d) be a complete metric space endowed with a graph and T be self-map on X. Suppose that:

i. G is weakly connected;
ii. T is $\left(A_\varphi, \tilde{G}\right)$-contraction and orbitally G-continuous;
ii. X_T is nonempty.

Then T is a PO.

Proof. Let $x_0 \in X_T$, then Theorem 2 implies that $\{T^n x_0\}$ is Cauchy sequence. Owing to completeness of X, $\{T^n x_0\}$ is converges to $x^* \in X$. Because $\left(T^n x_0, T^{n+1}x_0\right) \in E\left(G\right)$ for all $n \in \mathbb{N}$ and T is orbitally G-continuous, therefore $x^* = \lim_{n\to\infty} T\left(T^n x_0\right) = Tx^*$. That is, $Tx^* = x^*$. Assume that y^* is another fixed point of T. If we use the same technique as in the Theorem 2, then we obtain that $y^* = x^*$.

Corollary 1. Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be an edge-preserving and the set X_T be nonempty. Suppose that:

i. G is weakly connected and (C)-graph;
ii. there exist some $\alpha \in A$ such that

$$d\left(Tx, Ty\right) \leq \alpha\left(d\left(x, y\right), d\left(x, Tx\right), \left(y, Ty\right)\right)$$

for each $(x, y) \in E\left(\tilde{G}\right)$.

Then T is a PO.

Corollary 2. Let (X, d) be a complete metric space endowed with a graph G and $T : X \to X$ be an edge-preserving and the set X_T be nonempty. Suppose that:

i. G is weakly connected;
ii. T is orbitally G-continuous;
iii. there exist some $\alpha \in A$ such that

$$d\left(Tx, Ty\right) \leq \alpha\left(d\left(x, y\right), d\left(x, Tx\right), \left(y, Ty\right)\right)$$

for each $(x, y) \in E\left(\tilde{G}\right)$.

Then T is a PO.

The following example verify all conditions of Theorem 1, also since T is orbitally G-continuous satisfy all conditions of Theorem 2.

Example 1. Let $X = \{0, 1, 2, 3, 4\}$ and $d\left(x, y\right) = |x - y|$, for all $x, y \in X$. Consider

$$E\left(\tilde{G}\right) = \{(0,0), (1,1), (2,2), (3,3), (4,4), (0,1), (0,2), (0,3), (1,4), (3,4)\},$$

and $T : X \to X$ as follows:

$$Tx = \begin{cases} 0, & x \in \{2,3\} \\ 1, & x \in \{0,1,4\} \end{cases}.$$

Then G is weakly connected and (C)-graph, X_T is nonempty and T is a $\left(A_\varphi, \tilde{G}\right)$-contraction and orbitally G-continuous, where $\varphi(t) = \frac{t}{2}$. Moreover, 1 is a unique fixed point of T.

3. Common fixed point theorems on directed graphs

First, we introduce μ-graph. Also, we prove common fixed-point theorems in metric space with a directed graph by using this concept.

Definition 9. Let S, T be self-maps on X. We say that G is a μ-graph whenever for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$ and $(x_{2n}, x_{2n+1}) \in E(G)$ for all $n \geq 0$, there is a subsequence $\{x_{2n_p}\}$ of $\{x_{2n}\}$ such that either T is continuous and $\left(x, x_{2n_p+1}\right) \in E(G)$ for all $p \geq 0$ or S is continuous and $\left(x_{2n_p}, x\right) \in E(G)$ for all $p \geq 0$.

Theorem 3. Let (X, d) be a complete metric space endowed with a directed graph G and S, T be self-maps on X. Assume that the following assertions hold:

i. G is a μ-graph;

ii. there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, such that

$$(x_{2n}, Sx_{2n}) \in E(G) \quad \Rightarrow \quad (x_{2n+2}, Sx_{2n+2}) \in E(G)$$

and

$$(x_{2n+1}, Tx_{2n+1}) \in E(G) \quad \Rightarrow \quad (x_{2n+3}, Tx_{2n+3}) \in E(G);$$

iii. there exist some $\alpha \in A_\varphi$ such that

$$d(Sx, Ty) \leq \alpha\left(d(x,y), d(x,Sx), (y,Ty)\right)$$

for each $(x, y) \in E(G)$;

iv. $X_{ST} = \{x \in X : (x, Sx) \in E(G) \text{ and } (Sx, TSx) \in E(G)\}$ is nonempty.

Then S, T have a common fixed point. Moreover, suppose that for any two common fixed point x^*, y^* of S and T, there exists $z \in X_p$ such that $(x^*, z) \in E(G)$ and $(z, y^*) \in E(G)$. Thus, S, T have a unique common fixed point.

Proof. Let x_0 be arbitrary element of X, and establish $\{x_n\}$ by

$$x_{2n+2} = Tx_{2n+1}$$

$$x_{2n+1} = Sx_{2n}, \quad n = 0, 1, 2, \ldots \quad .$$

Let $(x_0, Sx_0) \in E(G)$ and $(x_1, Tx_1) \in E(G)$, for $x_0, x_1 \in X$. Then from the assumption for $x_2, x_3 \in X$, $(x_2, Sx_2) \in E(G)$ and $(x_3, Tx_3) \in E(G)$. Then we continuing this way, $(x_{2n}, Sx_{2n}) \in E(G)$ and $(x_{2n+1}, Tx_{2n+1}) \in E(G)$, for all $n \in \mathbb{N}$. Also, $(x_{2n}, x_{2n+1}) \in E(G)$ and $(x_{2n+1}, x_{2n+2}) \in E(G)$, for all $n \in \mathbb{N}$. From (iii), we get

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha\left(d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1})\right) \\ &= \alpha\left(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\right). \end{aligned}$$

by definition of α,

$$d(x_{2n+1}, x_{2n+2}) \le \varphi(d(x_{2n}, x_{2n+1})),$$

for all $n \in \mathbb{N}$. Similarly, from (ii), $(x_{2n}, Sx_{2n}) \in E(G)$ and so $(x_{2n+2}, Sx_{2n+2}) \in E(G)$. That is, $(x_{2n+2}, x_{2n+3}) \in E(G)$. Thus, by using (iii),

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(Sx_{2n+2}, Tx_{2n+1}) \\ &\le \alpha(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, Sx_{2n+2}), d(x_{2n+1}, Tx_{2n+1})) \\ &= \alpha(d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2})) \end{aligned}$$

by definition of α,

$$d(x_{2n+2}, x_{2n+3}) \le \varphi(d(x_{2n+1}, x_{2n+2})),$$

for all $n \in \mathbb{N}$. By this way, if we continue, we get

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &\le \varphi(d(x_{2n+1}, x_{2n+2})) \\ &\le \varphi(\varphi(d(x_{2n}, x_{2n+1}))) \\ &= \varphi^2(d(x_{2n}, x_{2n+1})) \\ &\vdots \\ &\le \varphi^{2n+2}(d(x_0, x_1)), \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $d(x_n, x_{n+1}) \le \varphi^n(d(x_0, x_1))$, for all $n \in \mathbb{N}$. Since from definition 6 (ii), we have $\lim_{n\to\infty} \varphi^n(d(x_0, x_1)) = 0$. For a given $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \ge n_0$.

$$\varphi^n(d(x_0, x_1)) < \varepsilon - \varphi(\varepsilon).$$

Hence

$$d(x_n, x_{n+1}) < \varepsilon - \varphi(\varepsilon), \tag{3.1}$$

for all $n \ge n_0$.
Now, for any $m, n \in \mathbb{N}$ with $m > n \ge n_0$, we claim that

$$d(x_n, x_m) < \varepsilon. \tag{3.2}$$

We prove the inequality (3.2) by induction on m. The inequality (3.2) holds for $m = n+1$, by using (3.1). Assume that (3.2) holds for $m = k$. So that $m = k+1$, we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{k+1}) \\ &< \varepsilon - \varphi(\varepsilon) + d(x_{n+1}, x_{k+1}) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(d(x_n, x_k)) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

By induction on m, we conclude that (3.2) holds for $m > n \geq n_0$. Thus, $\{x_n\}$ is a Cauchy sequence in X. Because X is complete, $x_n \to x^* \in X$.

Now we prove that x^* is a common fixed point of S and T. Because of $x_n \to x$ and $(x_{2n}, x_{2n+1}) \in E(G)$, because G is μ-graph, there exists a subsequence $\{x_{2n_p}\}$ of $\{x_{2n}\}$ such that either T is continuous and $\left(x^*, x_{2n_p+1}\right) \in E(G)$ or S is continuous and $\left(x_{2n_p}, x^*\right) \in E(G)$. Assume that T is continuous and $\left(x^*, x_{2n_p+1}\right) \in E(G)$. thus

$$x^* = \lim_{n\to\infty} x_{2n_p+1} \quad \Rightarrow \quad Tx^* = \lim_{n\to\infty} Tx_{2n_p+1} = \lim_{n\to\infty} x_{2n_p+2} = x^*.$$

That is, $Tx^* = x^*$. Moreover from (iii), we get

$$d\left(Sx^*, Tx_{2n_p+1}\right) \leq \alpha\left(d\left(x^*, x_{2n_p+1}\right), d\left(x^*, Sx^*\right), d\left(x_{2n_p+1}, Tx_{2n_p+1}\right)\right),$$

as $p \to \infty$,

$$d\left(Sx^*, x^*\right) \leq \alpha\left(0, d\left(x^*, Sx^*\right), 0\right)$$

$$\leq \varphi(0) = 0.$$

Then $Sx^* = x^*$. Thus, x^* is a common fixed point of S and T. Similarly, suppose that S is continuous and $\left(x_{2n_p}, x^*\right) \in E(G)$, we obtain again desired result.

For the uniqueness, y^* is another common fixed point of S and T. Then $(x^*, z) \in E(G)$ and $(z, y^*) \in E(G)$. As G is a directed graph, so $(x^*, y^*) \in E(G)$.

$$d\left(x^*, y^*\right) = d\left(Sx^*, Ty^*\right) \leq \alpha\left(d\left(x^*, y^*\right), d\left(x^*, Sx^*\right), d\left(y^*, Ty^*\right)\right)$$

$$\leq \alpha\left(d\left(x^*, y^*\right), 0, 0\right)$$

$$\leq \varphi(0) = 0,$$

which is a contradiction. Therefore, x^* is unique common fixed point of S and T.

Corollary 3. Let (X, d) be a complete metric space endowed with a directed graph G and S, T be self-maps on X. Assume that the following assertions hold:

i. G is a μ-graph;

ii. there is a sequence $\{x_n\}_{n\in\mathbf{N}}$ in X, such that

$$\left(x_{2n}, Sx_{2n}\right) \in E(G) \quad \Rightarrow \quad \left(x_{2n+2}, Sx_{2n+2}\right) \in E(G)$$

and

$$\left(x_{2n+1}, Tx_{2n+1}\right) \in E(G) \quad \Rightarrow \quad \left(x_{2n+3}, Tx_{2n+3}\right) \in E(G);$$

iii. there exist some $\alpha \in A$ such that

$$d(Sx, Ty) \leq \alpha\left(d(x, y), d(x, Sx), (y, Ty)\right)$$

for each $(x, y) \in E(G)$;

iv. X_{ST} is nonempty.

Then S, T have a common fixed point.

The following example demonstrate that all conditions of Theorem 3 are satisfied.

Example 2. Let $X = \{0, 1, 2, 3,\}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Consider

$$E(G) = \{(0,0), (0,2), (2,0), (1,1), (1,2), (3,1), (0,3)\},$$

and $S, T : X \to X$ as follows:

$$Sx = \begin{cases} 1, & x \in \{1,3\} \\ 2, & x = 0 \\ 3, & x = 2 \end{cases} \qquad Tx = \begin{cases} 0, & x = 3 \\ 1, & x \in \{1,2\} \\ 2, & x = 0. \end{cases}$$

Then all conditions of Theorem 3 is satisfied with $\varphi(t) = \frac{2t}{3}$. Moreover, 1 is a unique common fixed point of S and T.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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