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Common fixed point theorems for hybrid F-contractions and applications

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Abstract In the present paper, we derive a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying closed multi-valued F-contraction condition introduced by Wardowski [Fixed Point Theory Appl. 2012:94] via common limit range property in the frame of complete metric spaces. Also, hybrid mappings which satisfy an F-contractive condition of Hardy-Rogers type are considered. Our results improve several results from the existing literature. Two applications are presented—the proofs of existence of solutions for certain system of functional equations arising in dynamic programming, as well as for certain Volterra integral inclusion.

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1. Introduction and preliminaries

The development of geometric fixed point theory for multivalued mappings was initiated with the work of Nadler, Jr. [28] in the year 1969. He used the concept of Hausdorff metric to establish the multivalued contraction principle containing Banach Contraction Principle as a special case, as follows.

Theorem 1.1. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a mapping from $\mathcal{X}$ into $CB(\mathcal{X})$ such that for all $x, y \in \mathcal{X}$,*

$$\mathcal{H}(\mathcal{T}x, \mathcal{T}y) \leq \lambda d(x, y),$$

where $\lambda \in [0, 1)$. Then $\mathcal{T}$ has a fixed point, that is, there exists a point $x \in \mathcal{X}$ such that $x \in \mathcal{T}x$.

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Hybrid fixed point theory for pairs of single-valued and multivalued mappings is a new development in Nonlinear Analysis (see, e.g., [11, 12, 15, 24, 29, 45] and references therein). The concepts of commutativity and weak commutativity were extended to multivalued mappings on metric spaces by Kaneko [20, 21]. In 1989, Singh et al. [40] extended the notion of compatible mappings and obtained some coincidence and common fixed point theorems for nonlinear hybrid contractions. It was observed that under compatibility the fixed point results usually require continuity of one of the underlying mappings. Afterwards, Pathak [30] generalized the concept of compatibility by defining weak compatibility for hybrid pairs of mappings (including single valued case) and utilized it to prove common fixed point theorems. Naturally, compatible mappings are weakly compatible but not conversely.

Kamran [19] extended the notion of property (E.A) (previously introduced for single-valued mappings by Aamri and Moutawakil [1]) to hybrid pairs of mappings. In 2011, Sintunavarat and Kumam [44] introduced the notion of common limit range property for single-valued mappings and showed its superiority over property (E.A). Motivated by this fact, Imdad et al. [14] established common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric (semi-metric) spaces.

For an extensive collection of hybrid contraction conditions, we refer to [2, 7, 10, 13, 16, 18, 22, 29, 34, 35, 41–43].

The following definitions and results are standard in the theory of multivalued and hybrid mappings.

Let (X, d) be a metric space. Then, on the lines of Nadler [28], we adopt that:

(1) $CL(X) = \{A : A$ is a non-empty closed subset of $X\}$,
(2) $CB(X) = \{A : A$ is a non-empty closed and bounded subset of $X\}$,
(3) For non-empty closed and bounded subsets A, B of X and $x \in X$,

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

and

$$\mathcal{H}(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$

It is well known that $CB(X)$ is a metric space with the distance $\mathcal{H}$ which is known as the Hausdorff-Pompeiu metric on $CB(X)$.

Let $f : X \to X$ and $T : X \to CB(X)$.

(1) A point $x \in X$ is a fixed point of f (resp. T) if $x = fx$ (resp. $x \in Tx$). The set of all fixed points of f (resp. T) is denoted by $F(f)$ (resp. $F(T)$).
(2) A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $C(f, T)$.
(3) A point $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and T is denoted by $F(f, T)$.
(4) For a mapping $T : X \to CL(X)$, the graph of T is the subset $G(T) = \{(x, y) : x \in X, y \in Tx\}$ of $X \times X$. Then T is a closed multivalued mapping if $G(T)$ is a closed subset of $X \times X$.

We also recall the following terminology, usually used for hybrid pairs of mappings.

Definition 1.2. *Let (X, d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$. A hybrid pair of mappings (f, T) is said to be:*

(1) commuting on X [20] if $fTx \subseteq Tfx$ for all $x \in X$.
(2) weakly commuting on X [21] if $\mathcal{H}(fTx, Tfx) \leq d(fx, Tx)$ for all $x \in X$.

(3) compatible [40] if $fTx \in CB(X)$ *for all* $x \in X$ *and* $\lim_{n\to\infty} \mathcal{H}(Tfx_n, fTx_n) = 0$*, whenever* $\{x_n\}$ *is a sequence in* X *such that* $Tx_n \to A \in CB(X)$ *and* $fx_n \to t \in A$*, as* $n \to \infty$*.*
(4) non-compatible [22] if there exists at least one sequence $\{x_n\}$ *in* X *such that* $Tx_n \to A \in CB(X)$ *and* $fx_n \to t \in A$*, as* $n \to \infty$ *but* $\lim_{n\to\infty} \mathcal{H}(Tfx_n, fTx_n)$ *is either non-zero or nonexistent.*
(5) weakly compatible [17] if $fTx = Tfx$ *for each* $x \in C(f,T)$*.*
(6) coincidentally idempotent [13] if $ffv = fv$ *for every* $v \in C(f,T)$*, i.e.,* f *is idempotent at the coincidence points of* f *and* T*.*
(7) occasionally coincidentally idempotent [36] if $ffv = fv$ *for some* $v \in C(f,T)$*.*
(8) with the property (E.A) [19] if there exists a sequence $\{x_n\}$ *in* X *such that*

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n,$$

for some $t \in X$ *and* $A \in CB(X)$*.*
(9) with common limit range property with respect to the mapping f *[14] if there exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$ *and* $A \in CB(X)$*.*

The following easy example shows the relationship of the occasionally coincidentally idempotent notion to other conditions of the previous definition.

Example 1.3. *[18, Example 1] Let* $X = \{1,2,3\}$ *(with the standard metric),*

$$f : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad T : \begin{pmatrix} 1 & 2 & 3 \\ \{1\} & \{1,3\} & \{1,3\} \end{pmatrix}.$$

Then it is easy to see the following:

- $C(f,T) = \{1,2\}$*, and* $F(f,T) = \{1\}$*.*
- (f,T) *is not commuting and not weakly commuting.*
- (f,T) *is not compatible.*
- (f,T) *is not weakly compatible.*
- (f,T) *is not coincidentally idempotent since* $ff2 = f3 = 2 \neq 3 = f2$*.*
- (f,T) *is occasionally coincidentally idempotent since* $ff1 = 1 = f1$*.*

Obviously, in this case (f,T) *is also non-compatible, but simple modifications of this example can show that the occasionally coincidentally idempotent property is independent of this notion, too.*

The following examples, also taken from [18], demonstrate the relationship between (E.A) property and common limit range property.

Example 1.4. *[18, Examples 2 and 3] Let* $X = [0, 2]$ *be equipped with the usual metric* $d(x, y) = |x - y|$. *Define* $f, g : X \to X$ *and* $T : X \to CB(X)$ *as follows:*

$$fx = \begin{cases} 2 - x, & \text{if } 0 \le x < 1, \\ \frac{9}{5}, & \text{if } 1 \le x \le 2; \end{cases}$$

$$gx = \begin{cases} 2 - x, & \text{if } 0 \le x \le 1, \\ \frac{9}{5}, & \text{if } 1 < x \le 2; \end{cases}$$

$$Tx = \begin{cases} \left[\frac{1}{2}, \frac{3}{2}\right], & \text{if } 0 \le x \le 1, \\ \left[\frac{1}{4}, \frac{1}{2}\right], & \text{if } 1 < x \le 2. \end{cases}$$

One can verify that the pair (f, T) *enjoys the property (E.A), but not the common limit range property with respect to the mapping* f. *On the other hand, the pair* (g, T) *satisfies the common limit range property with respect to the mapping* g.

Remark 1.5. *Note that if a pair* (f, T) *satisfies the property (E.A) along with the closedness of* $f(X)$, *then the pair also satisfies the common limit range property with respect to the mapping* f.

In the present paper, we first derive a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying closed multi-valued F-contraction condition, via common limit range property in the framework of complete metric spaces. Also, a variant of F-contractive condition of Hardy-Rogers type is utilized. We note that such conditions were introduced for multivalued mappings by Sgroi and Vetro [39], who modified previously introduced notions used by Wardowski [47] for single-valued mappings. Our results improve several results from the existing literature. Two applications are presented—the proofs of existence of solutions for certain system of functional equations arising in dynamic programming, as well as for certain Volterra integral inclusion.

2. Common fixed point theorems for hybrid mappings

The attempted improvements in this paper are four-fold.

(i) occasionally coincidentally idempotent notion is used, which is weaker than coincidentally idempotent in the case when the set of coincidence points is not empty, see Example 1.3;

(ii) common limit range property is utilized (instead of property (E.A));

(iii) any requirement of closedness of the range of f is relaxed;

(iv) a new kind of contractive conditions (so-called F-contraction conditions) are used, that originated in the work of Wardowski [47].

This section is divided into two parts. In the first subsection, we prove a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying closed multi-valued F-contractions condition via common limit range property in the framework of complete metric spaces, while in the second one we obtain results for hybrid pairs which satisfy an F-contractive condition of Hardy-Rogers-type.

To complete the results, we use the following notions.

Throughout the article we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers. In what follows, $\mathcal{F}$ will denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions

(F1) F is continuous and strictly increasing;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty}\alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha) = 0$.

Some examples of functions $F \in \mathcal{F}$ are $F(t) = \ln t$, $F(t) = t + \ln t$, $F(t) = -1/\sqrt{t}$, see [47].

Definition 2.1. *[47] Let (X, d) be a metric space. A self-mapping T on X is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(Tx, Ty)) \le F(d(x, y)), \tag{2.1}$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$.

Example 2.2. *[47] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(x) = \ln x$. It is clear that F satisfies (F1)–(F3) for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfying (2.1) is an F-contraction such that*

$$d(Tx, Ty) \le e^{-\tau} d(x, y), \text{ for all } x, y \in X, \quad Tx \ne Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the previous inequality also holds and hence T is a contraction.

In what follows, for a metric space (X, d) and a multivalued mapping $T : X \to CL(X)$, we will denote

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

Definition 2.3. *[39] Let (X, d) be a metric space. A multivalued mapping $T : X \to CL(X)$ is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x, y \in X$ with $y \in Tx$ there exists $z \in Ty$ for which*

$$\tau + F(d(y, z)) \le F(M(x, y)). \tag{2.2}$$

if $d(y, z) > 0$.

Example 2.4. *[39] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(x) = \ln x$. For each multivalued mapping $T : X \to CL(X)$ satisfying (2.2) we have*

$$d(y, z) \le e^{-\tau} M(x, y), \text{ for all } x, y \in X, \quad z \in Ty, \ y \ne z.$$

It is clear that for $z, y \in X$ such that $y = z$ the previous inequality also holds.

Some fixed point results for single-valued, resp. multivalued F-contractions were obtained in [3, 23, 47], resp. [39].

2.1. Result - I

Our first main result is as follows:

Theorem 2.5. *Let (X, d) be a metric space and let $f : X \to X$ and $T : X \to CB(X)$. Assume that there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\begin{aligned}&\tau + F(\mathcal{H}(Tx, Ty))\\ &\quad\le F\left(\max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}\left[d(fx, Ty) + d(fy, Tx)\right]\right\}\right),\end{aligned} \tag{2.3}$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Ty) > 0$. Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point.

If, moreover, the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.

Proof. Since the pair (f, T) satisfies the common limit range property with respect to the mapping f, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$ and $A \in CB(X)$. We assert that $fu \in Tu$. If not, then using condition (2.3), we get

$$\begin{aligned}&\tau + F(\mathcal{H}(Tx_n, Tu))\\ &\quad\leq \quad F\left(\max\left\{d(fx_n, fu), d(fx_n, Tx_n), d(fu, Tu), \frac{1}{2}\left[d(fx_n, Tu) + d(fu, Tx_n)\right]\right\}\right).\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\tau + F(\mathcal{H}(A, Tu)) \leq F\left(\max\left\{0, d(fu, A), d(fu, Tu), \frac{1}{2}[d(fu, Tu) + d(fu, A)]\right\}\right).$$

Since $\tau > 0$ and F is strictly increasing, we have

$$\mathcal{H}(A, Tu) < \max\left\{0, 0, d(fu, Tu), \frac{1}{2}[d(fu, Tu) + 0]\right\} = d(fu, Tu).$$

Since $fu \in A$, the above inequality implies

$$d(fu, Tu) \leq \mathcal{H}(A, Tu) < d(fu, Tu),$$

which is a contradiction. Hence $fu \in Tu$ which shows that the pair (f, T) has a coincidence point (i.e., $\mathcal{C}(f, T) \neq \emptyset$).

If it is assumed that the mappings f and T are occasionally coincidentally idempotent, then $ffv = fv$ for some $v \in \mathcal{C}(f, T)$ which implies $ffv = fv \in Tv$. Now we show that $Tv = Tfv$. If not, using condition (2.3), we get

$$\begin{aligned}&\tau + F(\mathcal{H}(Tfv, Tv))\\ &\leq F\left(\max\left\{d(ffv, fv), d(ffv, Tfv), d(fv, Tv), \frac{1}{2}\left[d(ffv, Tv) + d(fv, Tfv)\right]\right\}\right)\\ &= F\left(\max\left\{0, d(fv, Tfv), d(fv, Tv), \frac{1}{2}\left[d(fv, Tv) + d(fv, Tfv)\right]\right\}\right).\end{aligned}$$

Since $fv \in Tv$, the above inequality implies

$$\begin{aligned}\tau + F(\mathcal{H}(Tfv, Tv)) &\leq F\left(\max\left\{d(fv, Tfv), \frac{1}{2}d(fv, Tfv)\right\}\right)\\ &= F(d(Tfv, fv)).\end{aligned}$$

Using $(F1)$, we get

$$F(d(Tfv, fv)) < F(d(Tfv, fv)),$$

which is a contradiction. Thus we have $fv = ffv \in Tv = Tfv$ which shows that fv is a common fixed point of the mappings f and T. ∎

We note that in a metric space every upper semicontinuous multi-valued mapping is closed (see [39, p. 1262]). Then, from Theorem 2.5, we obtain the following corollary.

Corollaries 2.6. *Let (X, d) be a metric space and $f : X \to X$. Let $T : X \to CB(X)$ be upper semicontinuous. Assume that there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Ty)) \\ \leq F\left(\max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \tfrac{1}{2}\left[d(fx, Ty) + d(fy, Tx)\right]\right\}\right),$$

for all $x, y \in X$ with $\mathcal{H}(Tx, Ty) > 0$. Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point.

If, moreover, the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.

In view of Remark 1.5, we have the following natural result.

Corollaries 2.7. *Let f be a self mapping of a metric space (X, d) and T be a mapping from X into $CB(X)$ satisfying condition (2.3) of Theorem 2.5. Suppose that the pair (f, T) satisfies the property (E.A) along with the closedness of $f(X)$. Then the mappings f and T have a coincidence point. Moreover, if the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.*

Notice that a non-compatible hybrid pair always satisfies the property (E.A). Hence, we get the following corollary.

Corollaries 2.8. *Let f be a self mapping of a metric space (X, d) and T be a mapping from X into $CB(X)$ satisfying condition (2.3) of Theorem 2.5. Suppose that the pair (f, T) is non-compatible and $f(X)$ is a closed subset of X. Then the mappings f and T have a coincidence point. Moreover, if the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.*

2.2. Result - II

Our second main result is as follows:

Theorem 2.9. *Let (X, d) be a metric space. Let $f : X \to X$ and $T : X \to CB(X)$. Assume that there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}(Tx, Ty)) \\ \leq F(\alpha\, d(fx, fy) + \beta\, d(fx, Tx) + \gamma\, d(fy, Ty) + \delta\, d(fx, Ty) + \varepsilon\, d(fy, Tx)), \tag{2.4}$$

for all $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$, $\alpha + \beta + \gamma + \delta + \varepsilon \leq 1$. Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point.

If, moreover, the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.

Proof. Since the pair (f, T) satisfies the common limit range property with respect to the mapping f, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$ and $A \in CB(X)$. We assert that $fu \in Tu$. If not, then using condition (2.4), we get

$$\begin{aligned}\tau &+ F(\mathcal{H}(Tx_n, Tu)) \\ &\leq F(\alpha\, d(fx_n, fu) + \beta\, d(fx_n, Tx_n) + \gamma\, d(fu, Tu) + \delta\, d(fx_n, Tu) + \varepsilon\, d(fu, Tx_n)).\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\tau + F(\mathcal{H}(A, Tu)) \leq F((\gamma + \delta)d(fu, Tu)),$$

Since $\tau > 0$ and F is strictly increasing, it follows that

$$d(fu, Tu) \leq d(A, Tu) < (\gamma + \delta)d(fu, Tu).$$

But, since $\gamma + \delta \leq 1$, this is a contradiction. Hence, $fu \in Tu$ which shows that the pair (f, T) has a coincidence point (i.e., $\mathcal{C}(f, T) \neq \emptyset$).

If it is assumed that the mappings f and T are occasionally coincidentally idempotent, then $ffv = fv$ for some $v \in \mathcal{C}(f, T)$ which implies $ffv = fv \in Tv$. Now we show that $Tv = Tfv$. If not, using condition (2.4), we get

$$\begin{aligned}&\tau + F(\mathcal{H}(Tfv, Tv)) \\ &\leq F(\alpha\, d(ffv, fv) + \beta\, d(ffv, Tfv) + \gamma\, d(fv, Tv) + \delta\, d(ffv, Tv) + \varepsilon\, d(fv, Tfv)) \\ &= F(\beta\, d(fv, Tfv) + \gamma\, d(fv, Tv) + \delta\, d(fv, Tv) + \varepsilon\, d(fv, Tfv)).\end{aligned}$$

Since $fv \in Tv$, the above inequality implies

$$\tau + F(d(Tfv, Tv)) \leq F((\beta + \varepsilon)d(fv, Tfv)).$$

Using $(F1)$, we get

$$d(Tfv, fv) < (\beta + \varepsilon)d(fv, Tfv),$$

which is a contradiction, as $\beta + \varepsilon \leq 1$. Thus we have $fv = ffv \in Tv = Tfv$ which shows that fv is a common fixed point of the mappings f and T. ∎

Similarly to Corollary 2.6, we have the following consequence of Theorem 2.9:

Corollaries 2.10. *Let (X, d) be a metric space and $f : X \to X$. Let $T : X \to CB(X)$ be upper semicontinuous. Assume that there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\begin{aligned}\tau &+ F(\mathcal{H}(Tx, Ty)) \\ &\leq F(\alpha\, d(fx, fy) + \beta\, d(fx, Tx) + \gamma\, d(fy, Ty) + \delta\, d(fx, Ty) + \varepsilon d(fy, Tx)),\end{aligned}$$

for all $x, y \in X$ with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, \varepsilon \geq 0$, $\alpha + \beta + \gamma + \delta + \varepsilon \leq 1$. Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point.

If, moreover, the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.

If $F : \mathbb{R}^+ \to \mathbb{R}$ is given by $F(x) = \ln x$ and denoting $e^{-\tau} = k$, then we have the following existing known results for some classes of contractions:

Corollaries 2.11. *Let (X, d) be a metric space. Let $f : X \to X$ and $T : X \to CB(X)$. Assume that there exists $k \in (0, 1)$ such that*

$$\mathcal{H}(Tx, Ty)) \leq k \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}\left[d(fx, Ty) + d(fy, Tx)\right]\right\},$$

Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point.

If, moreover, the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.

Corollaries 2.12. *Let (X, d) be a metric space. Let $f : X \to X$ and $T : X \to CB(X)$. Assume that there exists $k \in (0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \le k(\alpha\, d(fx, fy) + \beta\, d(fx, Tx) + \gamma\, d(fy, Ty) + \delta\, d(fx, Ty) + \varepsilon\, d(fy, Tx)),$$

for all $x, y \in X$ with $Tx \ne Ty$, where $\alpha, \beta, \gamma, \delta, \varepsilon \ge 0$, $\alpha + \beta + \gamma + \delta + \varepsilon \le 1$. Suppose that the pair (f, T) enjoys the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point.

If, moreover, the pair (f, T) is occasionally coincidentally idempotent, then the pair (f, T) has a common fixed point.

3. Applications to existence theorems for functional equations

The existence, uniqueness, and iterative approximations of solutions for several classes of functional equations arising in dynamic programming were studied by a lot of researchers. Bellman and Lee [5] first studied the existence of solutions for such functional equations. They pointed out that the basic form of functional equations in dynamic programming is as follows:

$$q(x) = \sup_{y \in D}\{G(x, y, q(\tau(x, y)))\}, \quad x \in W,$$

where $\tau : W \times D \to W$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings, while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, and U, V are Banach spaces.

In 1984, Bhakta and Mitra [6] obtained some existence theorems for the following functional equation which arises in multistage decision process related to dynamic programming

$$q(x) = \sup_{y \in D}\{g(x, y) + G(x, y, q(\tau(x, y)))\}, \quad x \in W,$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are mappings, while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, and U, V are Banach spaces.

Thereafter a lot of work have been done in this direction and existence and uniqueness results have been obtained for solutions and common solutions of some functional equations, as well as systems of functional equations in dynamic programming with the use of fixed point results. For details see [26, 27, 31–33, 37] and the references therein.

Consider now a multistage process, reduced to the system of functional equations

$$q_i(x) = \sup_{y \in D}\{g(x, y) + G_i(x, y, q_i(\tau(x, y)))\}, \quad x \in W, \quad i \in \{1, 2\}, \tag{3.1}$$

where $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, $G_i : W \times D \times \mathbb{R} \to \mathbb{R}$ are given mappings, while $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, and U, V are Banach spaces. The purpose of this section is to give conditions for the existence of solutions for the system of functional equations (3.1) using Theorem 2.5.

Let $B(W)$ be the set of all bounded real-valued functions on W, for an arbitrary $h \in B(W)$ define $\|h\| = \sup_{x \in W} |h(x)|$, and denote by d the respective metric. $(B(W), \|\cdot\|)$ is a Banach space and the respective convergence is uniform. Therefore, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then the sequence $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded. Therefore $h^* \in B(W)$.

We will consider the operators $T_i : B(W) \to B(W)$ given by

$$T_i h(x) = \sup_{y \in D} \{g(x, y) + G_i(x, y, h_i(\tau(x, y)))\}, \tag{3.2}$$

for $h \in B(W)$, $x \in W$, where $i \in \{1, 2\}$; these mappings are well-defined if the functions g and G_i are bounded. Also, denote

$$M(h, k) = \max\left\{d(T_2h, T_2k), d(T_2h, T_1h), d(T_2k, T_1k), \frac{d(T_1h, T_2k) + d(T_1k, T_2h)}{2}\right\},$$

for $h, k \in B(W)$.

Now, we are equipped to state and prove the following result.

Theorem 3.1. *Let $T_i : B(W) \to B(W)$ be given by (3.2), where $i \in \{1, 2\}$. Suppose that the following hypotheses hold:*

(1) there exist $\tau \in \mathbb{R}^+$ such that

$$|G_1(x, y, h(x)) - G_2(x, y, k(x))| \leq e^{-\tau} M(h, k)$$

for all $x \in W$, $y \in D$;

(2) $g : W \times D \to \mathbb{R}$ and $G_i : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions, where $i \in \{1, 2\}$;

(3) there exists a sequence $\{h_n\}$ in $B(W)$ and a function $h^* \in B(W)$ such that

$$\lim_{n\to\infty} T_1h_n = \lim_{n\to\infty} T_2h_n = T_1h^*;$$

(4) $T_1T_1h = T_1h$, whenever $T_1h = T_2h$, for some $h \in B(W)$.

Then the system of functional equations (3.1) has a bounded solution.

Proof. By hypothesis (3) the pairs (T_1, T_2) share the common limit range property with respect to T_1. Now, let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$. Then there exist $y_1, y_2 \in D$ such that

$$T_1h_1(x) < g(x, y_1) + G_1(x, y_1, h_1(\tau(x, y_1))) + \lambda, \tag{3.3}$$
$$T_1h_2(x) < g(x, y_2) + G_2(x, y_2, h_2(\tau(x, y_2))) + \lambda, \tag{3.4}$$
$$T_1h_1(x) \geq g(x, y_2) + G_1(x, y_2, h_1(\tau(x, y_2))), \tag{3.5}$$
$$T_1h_2(x) \geq g(x, y_1) + G_2(x, y_1, h_2(\tau(x, y_1))). \tag{3.6}$$

Next, by using (3.3) and (3.6), we obtain

$$\begin{aligned} T_1h_1(x) - T_1h_2(x) &< G_1(x, y_1, h_1(\tau(x, y_1))) - G_2(x, y_1, h_2(\tau(x, y_1))) + \lambda \\ &\leq |G_1(x, y_1, h_1(\tau(x, y_1))) - G_2(x, y_1, h_2(\tau(x, y_1)))| + \lambda \\ &\leq e^{-\tau} M(h_1, h_2) + \lambda \end{aligned}$$

and so we have

$$T_1h_1(x) - T_2h_2(x) < e^{-\tau} M(h_1, h_2) + \lambda. \tag{3.7}$$

Analogously, by using (3.4) and (3.5), we get

$$T_1h_2(x) - T_1h_1(x) < e^{-\tau} M(h_1, h_2) + \lambda \tag{3.8}$$

Finally, from (3.7) and (3.8), we deduce

$$|T_1h_1(x) - T_2h_2(x)| < e^{-\tau} M(h_1, h_2) + \lambda,$$

implying that

$$d(T_1h_1, T_2h_2) \le e^{-\tau}M(h_1, h_2) + \lambda.$$

Notice that the last inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, therefore we obtain that

$$d(T_1h_1, T_2h_2) \le e^{-\tau}M(h_1, h_2).$$

By passing to logarithms, we can write this as

$$\tau + \ln(d(T_1h_1, T_2h_2)) \le \ln(M(h_1, h_2)).$$

If we consider $F \in \mathcal{F}$ defined by $F(t) = \ln t$, for each $t \in (0, +\infty)$, and put $f = T_1$, $T = T_2$, the conditions of Theorem 2.5 are satisfied for the pair (f, T). Moreover, in view of the hypotheses (4), the pair (T_1, T_2) is occasionally coincidentally idempotent, and so T_1 and T_2 have a common fixed point, that is, the system of functional equations (3.1) has a bounded solution. ∎

4. Application to Volterra integral inclusions

Numerous studies have considered the Hammerstein integral inclusions or the Volterra integral inclusions that arise in the study of problems in applied mathematics, engineering and economics, since some mathematical models utilize multivalued maps instead of single-valued maps, see, e.g., [4, 8, 9, 38, 46] and references cited therein.

In this section we present another example where our Theorem 2.5 can be applied. This example is inspired by [46].

Here we establish new results for the existence of solutions of integral inclusion of the type

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s)F(s, x(s))\, ds \tag{4.1}$$

for $t \in J$, where $\sigma : J \to J$, $q : J \to E$, $k : J \times J \to \mathbb{R}$ are continuous and $F : J \times E \to C(E)$, where E is a Banach space with norm $\|\cdot\|_E$ and $C(E)$ denotes the class of all nonempty closed subsets of E, $J = [0, 1]$ in $\mathbb{R}$ is a closed and bounded interval.

By a solution for the integral inclusion (4.1), we mean a continuous function $x : J \to E$ such that

$$x(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v_1(s)\, ds$$

for some $v_1 \in B(J, E)$ satisfying $v_1(t) \in F(t, x(t))$, for all $t \in J$, where $B(J, E)$ is the space of all E-valued Bochner-integrable functions on J.

Let $C(J, E)$ denote the space of all continuous E-valued functions on J. Define a norm $\|\cdot\|$ on $C(J, E)$ by

$$\|x\| = \sup_{t \in J} \|x(t)\|_E.$$

We also need the following definitions in what follows.

Definition 4.1. *A multivalued map $F : J \to 2^E$ is said to be measurable if for any $y \in E$, the function $t \mapsto d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.*

Definition 4.2. *A multivalued function $\beta : J \times E \to 2^E$ is called Carathéodory if*

(i) $t \mapsto (t, x)$ is measurable for each $x \in E$, and

(ii) $x \mapsto (t, x)$ is upper semicontinuous almost everywhere for $t \in J$.

Denote

$$\|F(t,x)\| = \sup\{\|u\|_E : u \in F(t,x)\}.$$

Definition 4.3. *A Carathéodory multifunction $F(t,x)$ is called L^1-Carathéodory if for every real number $r > 0$ there exists a function $h_r \in L^1(J,R)$ such that*

$$\|F(t,x)\| \le h_r(t) \text{ for almost every } t \in J$$

and for all $x \in E$ with $\|x\|_E \le r$.

Denote

$$S_F^1(x) = \{v \in B(J,E) : v(t) \in F(t,x(t)) \text{ a. e. } t \in J\}.$$

Lemma 4.4. *[25] If diam$(E) < \infty$ and $F : J \times E \to 2^E$ is L^1-Carathéodory, then $S_F^1(x) \neq \emptyset$ for each $x \in C(J,E)$.*

Lemma 4.5. *[46] Let E be a Banach space, F a Carathéodory multi-map with $S_F^1 \neq \emptyset$ and let $\mathcal{L} : L^1(J,E) \to C(J,E)$ be a continuous linear mapping. Then the operator $\mathcal{L} \circ S_F^1 : C(J,E) \to 2^{C(J,E)}$ is a closed graph operator on $C(J,E) \times C(J,E)$.*

Suppose that the following set of hypotheses hold:

(H_0) The function $k(t,s)$ is continuous and nonnegative on $J \times J$ with

$$e^{-\tau} = \sup_{t,s \in J} k(t,s)$$

for some $\tau \in \mathbb{R}^+$;

(H_1) the multivalued function $F(t,x)$ is Carathéodory;

(H_2) the multivalued function $F(t,x)$ is nondecreasing in x almost everywhere for $t \in J$;

(H_3) there exists $\tau \in \mathbb{R}^+$ such that

$$|F(s,x(s)) - F(s,y(x))| \le e^{-\tau} M(x,y)$$

for all $s \in J$, $x \in E$;

(H_4) $S_F^1(x) \neq \emptyset$ for each $x \in C(J,E)$.

Theorem 4.6. *Assume that hypotheses (H_0)–(H_4) hold. Then the integral inclusion* (4.1) *has a solution in $[a,b]$ defined on J.*

Proof. Let $X = C(J,E)$ and consider the interval $[a,b]$ in X.

Define the multivalued mapping $T : [a,b] \to 2^X$ given for $u \in [a,b]$ by

$$Tx = \left\{ u : u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(s)\,ds, v \in S_F^1(x), \text{ for every } t \in [0,1] \right\}.$$

We point out that T is well-defined, since, from (H_4), $S_F^1(x) \neq \emptyset$. We shall show that T satisfies all conditions of Theorem 2.5 on $[a,b]$.

For all $\vartheta, \mu \in 2^X$ on $t \in [a,b]$ by (H_0) and (H_3) we get

$$\begin{aligned} \|\vartheta(t) - \mu(t)\|_E &= \left\| \int_0^{\sigma(t)} k(t,s)v_1(s)\,ds - \int_0^{\sigma(t)} k(t,s)v_2(s)\,ds \right\|_E \\ &\le \int_0^{\sigma(t)} k(t,s)\,ds \|v_1(s)ds - v_2(s)\|_E \\ &\le \sup_{t,s \in J} k(t,s) M(v_1,v_2) \text{ for } v_1, v_2 \in S_F^1(x). \end{aligned}$$

This implies that

$$\|\vartheta(t) - \mu(t)\| \leq e^{-\tau} M(v_1, v_2).$$

for each $t \in J$. Passing to logarithms, we can write this as

$$\tau + \|\vartheta(t) - \mu(t)\| \leq \ln(M(v_1, v_2)).$$

If we consider $F \in \mathcal{F}$ defined by $F(t) = \ln t$, for each $t \in (0, +\infty)$, we deduce that the operator T satisfy condition (2.3) where f is an identity mapping. Also T is a closed mapping. Using Theorem 2.5, we conclude that the given integral inclusion has a solution in $[a, b]$. ∎

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