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CONTROL OF CHAOS IN SPROTT SYSTEM B BY STATE SPACE EXACT LINEARIZATION METHOD

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Abstract In this communication, a non-linear chaotic dynamical system has been transferred into a linear controllable system using state space exact linearization method via lie algebra. Numerical simulation is applied to justify our claim.

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1. Introduction

In 1998, the state space exact linearization (SSEL) method for non-linear control system [4] is introduced by Liqun and Yanzhu [1]. They considered the very famous Lorenz dynamical system [5, 6] and applied their method. Here we consider Sprott's chaotic non-linear dynamical system B. This system has two non-linear terms with one non-zero

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parameter. To control the chaotic Sprott system B, we are applying the previously mentioned SSEL method via lie algebra, which transformed the non-linear dynamical system into a linear controllable system.

2. The Exact Linearization Method

Let us consider two smooth functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ which are also smooth vector fields on $\mathbb{R}^n$.
Let $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^1$ be the state variable and control parameter respectively. Then, any single-input non-linear system can be taken as

$$\dot{x} = f(x) + ug(x) \tag{2.1}$$

Let [X,Y] be the Lie bracket of vector fields X and Y. Let us denote

$$ad^i_f g(x) = [f, ad^{i-1}_f g](x), \quad i \geq 1 \tag{2.2}$$

with $ad^0_f g(x) = g(x)$.
If $f = [f_1, f_2, f_3, ..., f_n]^T$ and $g = [g_1, g_2, g_3, ..., g_n]^T$,
using Lie-algebra, we may consider

$$ad^i_f g = [(ad^i_f g)_1, (ad^i_f g)_2, (ad^i_f g)_3, ..., (ad^i_f g)_n]^T, \tag{2.3}$$

where

$$(ad^i_f g)_j = \sum_{k=1}^{n} \{f_k \frac{\partial(ad^{i-1}_f g)_j}{\partial x_k} - (ad^{i-1}_f g)_k \frac{\partial f_j}{\partial x_k}\}, \quad j = 1, 2, 3, ..., n \quad and \quad i \geq 1 \tag{2.4}$$

Let $L_X \lambda$ denote the Lie derivative of the real-valued function $\lambda(x)$ with respect to the vector field X.
Then, in a neighbourhood $N(x_0)$ of x_0 , one can easily obtain the relations

$$L_g \lambda(x) = L_{ad^1_f g} \lambda(x) = ... = L_{ad^{n-2}_f g} \lambda(x) = 0$$

and

$$L_{ad^{n-1}_f g} \lambda(x_0) \neq 0, \quad x \in N(x_0) \tag{2.5}$$

if the following two conditions are satisfied :
1) Rank of Δ = n, where $\Delta = [g(x_0), ad^1_f g(x_0), ..., ad^{n-1}_f g(x_0)]$ and
2) Γ is involutive , where $\Gamma = span\{g, ad^1_f g, ad^2_f g, ..., ad^{n-2}_f g\}$, near x_0.
Moreover, in the neighbourhood $N(x_0)$ of x_0, there exists the transformation

$$\begin{aligned} z &= [z_1, z_2, z_3, ..., z_n]^T \\ &= \Psi(x) \\ &= [\Psi_1(x), \Psi_2(x), \Psi_3(x), ..., \Psi_n(x)]^T \\ &= [\lambda(x), L_f \lambda(x), L^2_f \lambda(x), ..., L^{n-1}_f \lambda(x)]^T \end{aligned} \tag{2.6}$$

and

$$w = b(x) + ua(x) \tag{2.7}$$

where $a(x) = L_g L_f^{n-1}\lambda(x)$ and $b(x) = L_f^n \lambda(x)$.
Hence, we get the linear controllable system [2] from the non-linear system (2.1) by the SSEL method as given below :

$$\dot{z_1} = z_2, \quad \dot{z_2} = z_3, \quad \dot{z_3} = z_4, ..., \quad \dot{z_{n-1}} = z_n, \quad \dot{z_n} = w.$$

3. Control of the Sprott Model B

The Sprott's non-linear chaotical dynamical system [3] is given by

$$\begin{aligned} \dot{x_1} &= x_2x_3 \\ \dot{x_2} &= x_1 - x_2 \\ \dot{x_3} &= \gamma - x_1x_2 \end{aligned} \tag{3.1}$$

where $\gamma(\neq 0)$ is a parameter.
This system can be written as

$$\dot{x} = f(x) + ug(x)$$

where $f(x_1, x_2, x_3) = [x_2x_3, \quad x_1 - x_2, \quad \gamma - x_1x_2]^T, \quad x \in \mathbb{R}^3$,
$g(x_1, x_2, x_3) = [0, x_1, 0]^T, \quad u = u(x_1, x_2, x_3)$.
To make the linear controllable system using state space exact linearization method, we have to verify the two conditions as discussed earlier :
1) Rank of $\Delta = 3$, where $\Delta = [g, \quad ad_f^1 g, \quad ad_f^2 g]$ and
2) Γ is involutive , where $\Gamma = span\{g, \quad ad_f^1 g\}$.
Now, using (2.2), (2.3) and (2.4),

$$[f, g] = ad_f^1 g = [(ad_f^1 g)_1, \quad (ad_f^1 g)_2, \quad (ad_f^1 g)_3]^T \quad = [-x_1x_3, \quad x_2x_3 + x_1, \quad x_1^2]^T$$

Again, by similar process, we have,

$$[f, ad_f^1 g] = ad_f^2 g = \begin{bmatrix} -2x_2x_3^2 - x_1x_3 - \gamma x_1 \\ x_2x_3 + 2x_1x_3 + x_1 + \gamma x_2 - x_1x_2^2 \\ 2x_1x_2x_3 + x_1^2 \end{bmatrix}$$

So, if $x_1 \neq 0$,

$$|\Delta| = \begin{vmatrix} 0 & -x_1x_3 & -2x_2x_3^2 - x_1x_3 - \gamma x_1 \\ x_1 & x_2x_3 + x_1 & x_2x_3 + 2x_1x_3 + x_1 + \gamma x_2 - x_1x_2^2 \\ 0 & x_1^2 & 2x_1x_2x_3 + x_1^2 \end{vmatrix} = -\gamma x_1^4 \quad \neq 0,$$

$\therefore$ We can say, Rank of $\Delta = 3$.
Now, $(g, ad_f^1 g)_1 = 0, \quad (g, ad_f^1 g)_2 = 2x_1x_3, \quad (g, ad_f^1 g)_3 = 0$.
Hence, $[g, ad_f^1 g] = [0, \quad 2x_1x_3, \quad 0]^T$.
$\therefore$ it is similar like g as $[g, ad_f^1 g]$ is along the y-axis.
Thus, $\Gamma = span\{g, ad_f^1 g\}$ which is involutive, where $x_i \in \mathbb{R}, \quad i = 1, 2, 3$.
Hence, the required two conditions for the relation (2.5) have been satisfied for $n = 3$.
Now, $L_g\lambda(x) = 0$ gives $x_1 \frac{\partial \lambda}{\partial x_2} = 0$.
This implies that

$$\frac{\partial \lambda}{\partial x_2} = 0 \quad as \quad x_1 \neq 0 \tag{3.2}$$

$L_{ad^1_f g}\lambda(x) = 0$ gives

$$-x_1x_3\frac{\partial\lambda}{\partial x_1} + x_1^2\frac{\partial\lambda}{\partial x_3} = 0$$

$$or, \quad x_3\frac{\partial\lambda}{\partial x_1} - x_1\frac{\partial\lambda}{\partial x_3} = 0 \quad as \quad x_1 \neq 0 \tag{3.3}$$

Then, solution of the equation (3.3) is given by,

$$\lambda = x_1^2 + x_3^2 + \sigma \tag{3.4}$$

where σ is a numerical constant chosen in such way that the goal of the control can be attained.

Since $\frac{\partial\lambda}{\partial x_2} = 0$, using (3.3), we can write $L_{ad^2_f g}\lambda(x)$ as

$$\begin{aligned} L_{ad^2_f g}\lambda(x) &= (ad^2_f g)_1\frac{\partial\lambda}{\partial x_1} + (ad^2_f g)_3\frac{\partial\lambda}{\partial x_3} \\ &= -\frac{\gamma x_1^2}{x_3}\frac{\partial\lambda}{\partial x_3} \\ &= -2\gamma x_1^2 \\ &\neq 0, \quad since \quad x_1 \neq 0 \quad and \quad \gamma \neq 0 \end{aligned} \tag{3.5}$$

Using (3.4), we have,

$$L_f\lambda(x) = 2\gamma x_3 \tag{3.6}$$

and

$$\begin{aligned} L_f^2\lambda(x) &= L_f(L_f\lambda(x)) \\ &= x_2x_3\frac{\partial}{\partial x_1}(L_f\lambda(x)) + (x_1 - x_2)\frac{\partial}{\partial x_2}(L_f\lambda(x)) + (\gamma - x_1x_2)\frac{\partial}{\partial x_3}(L_f\lambda(x)) \\ &= 2\gamma(\gamma - x_1x_2) \end{aligned} \tag{3.7}$$

According to the state space exact linearization method, using (2.6) we can construct the transformation process as given below:

$$\begin{aligned} z &= [z_1, z_2, z_3]^T \\ &= \Psi(x) \\ &= [\Psi_1(x), \Psi_2(x), \Psi_3(x)]^T \\ &= [\lambda(x), L_f^1\lambda(x), L_f^2\lambda(x)]^T \end{aligned}$$

and

$$w = b(x) + ua(x) \tag{3.8}$$

where

$$a(x) = L_gL_f^2\lambda(x) \quad and \quad b(x) = L_f^3\lambda(x). \tag{3.9}$$

Hence,

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = z = \Psi(x) = \begin{bmatrix} x_1^2 + x_3^2 + \sigma \\ 2\gamma x_3 \\ 2\gamma(\gamma - x_1x_2) \end{bmatrix} \tag{3.10}$$

From (3.10), we can construct the inverse transformation as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x = \Psi^{-1}(z) = \begin{bmatrix} \frac{\sqrt{4\gamma^2 z_1 - 4\gamma^2\sigma - z_2^2}}{2\gamma} \\ \frac{2\gamma^2 - z_3}{\sqrt{4\gamma^2 z_1 - 4\gamma^2\sigma - z_2^2}} \\ \frac{z_2}{2\gamma} \end{bmatrix} \tag{3.11}$$

From (3.9), we have,

$$\begin{aligned} a(x) &= L_g L_f^2 \lambda(x) \\ &= x_1 \frac{\partial}{\partial x_2}(L_f^2 \lambda(x)) \\ &= -2\gamma x_1^2 \end{aligned} \tag{3.12}$$

$$\begin{aligned} b(x) &= L_f^3 \lambda(x) \\ &= x_2 x_3 \frac{\partial}{\partial x_1}(L_f^2\lambda(x)) + (x_1 - x_2)\frac{\partial}{\partial x_2}(L_f^2\lambda(x)) + (\gamma - x_1 x_2)\frac{\partial}{\partial x_3}(L_f^2\lambda(x)) \\ &= -2\gamma x_2^2 x_3 - 2\gamma x_1^2 + 2\gamma x_1 x_2 \end{aligned} \tag{3.13}$$

$\therefore$ From (3.8), u can be obtained as

$$u = \frac{w}{a(x)} - \frac{b(x)}{a(x)}$$

$$or, u = -\frac{w}{2\gamma x_1^2} - \frac{x_2^2 x_3 + x_1^2 - x_1 x_2}{x_1^2} \tag{3.14}$$

Using (3.8) and [(3.10) - (3.13)], we have got the linear controllable system from the system of equation (3.1) as given below:

$$\dot{z_1} = z_2, \quad \dot{z_2} = z_3 \quad and \quad \dot{z_3} = w \tag{3.15}$$

where w can be taken as

$$w = c_1 z_1 + c_2 z_2 + c_3 z_3 \tag{3.16}$$

in such way that by changing the value of σ, we can control the variable x_1 to the goal x_g.

Using (3.10) and (3.16) in (3.14), one gets,

$$u = -\frac{c_1(x_1^2 + x_3^2 + \sigma) + 2c_2\gamma x_3 + 2c_3\gamma(\gamma - x_1 x_2)}{2\gamma x_1^2} - \frac{x_2^2 x_3 + x_1^2 - x_1 x_2}{x_1^2} \tag{3.17}$$

which gives the linear feedback control parameter u.

With the help of (3.11) , we may consider the term σ as given below :

$$\sigma = -x_g^2 \tag{3.18}$$

where x_g is the control goal.

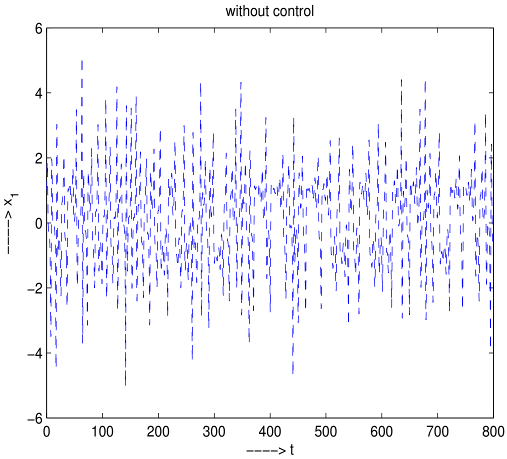

FIGURE 1. The time evolution of x_1 in uncontrolled system

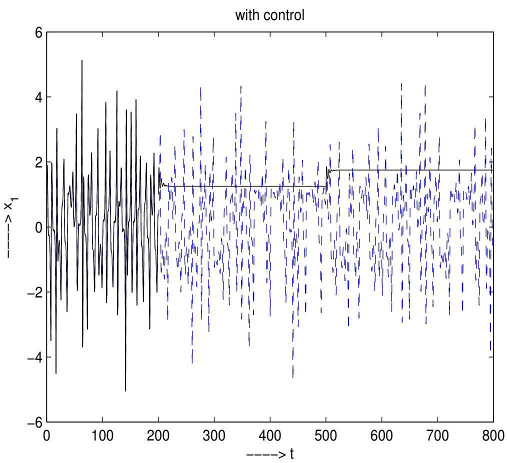

FIGURE 2. The time evolution of x_1 in controlled system

4. Results and Discussions

For $\gamma = 1$, Sprott found that the system (3.1) is chaotic. Numerical simulation is done with the initial values of the state variables $x_1(0) = 1, x_2(0) = 1$ & $x_3(0) = 1$. For the controlled system (3.15), the parameters c_1, c_2, c_3 are chosen[7] in such way that $c_1 + c_2c_3 > 0$ with $c_1 < 0, c_2 < 0$ & $c_3 < 0$ are satisfied.
Here, we have taken $c_1 = -0.5, c_2 = -2.5$ & $c_3 = -0.75$ and the control is started at $t = 200$ with $x_g = 1.25$. At time $t = 500$, the control goal is changed from $x_g = 1.25$ to $x_g = 1.75$.
Using Runge-Kutta method, time evolution of x_1, x_2, x_3 are shown in the figure(1), figure(3) and figure(5) respectively with $u = 0$ whereas figure(2), figure(4) and figure(6)

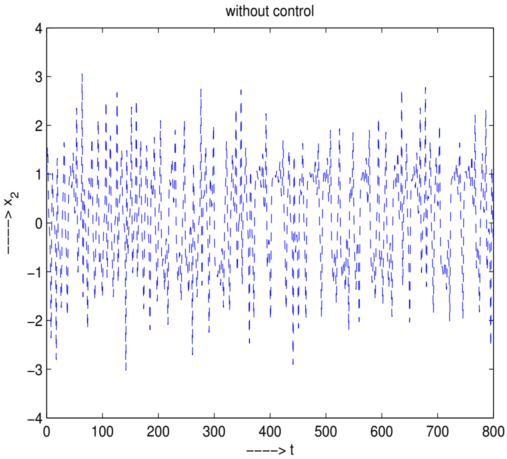

FIGURE 3. The time evolution of x_2 in uncontrolled system

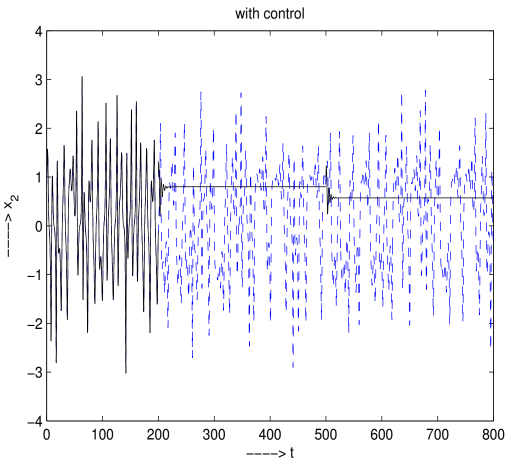

FIGURE 4. The time evolution of x_2 in controlled system

depict the time evolution of x_1, x_2, x_3 respectively in presence of control u.
Finally, we conclude that the SSEL method has been applied successfully to control the Sprott system B.

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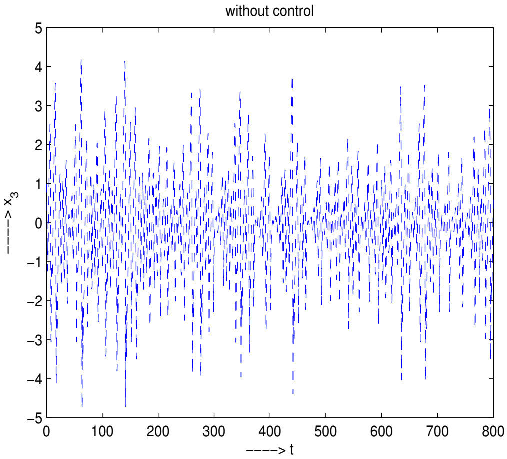

FIGURE 5. The time evolution of x_3 in uncontrolled system

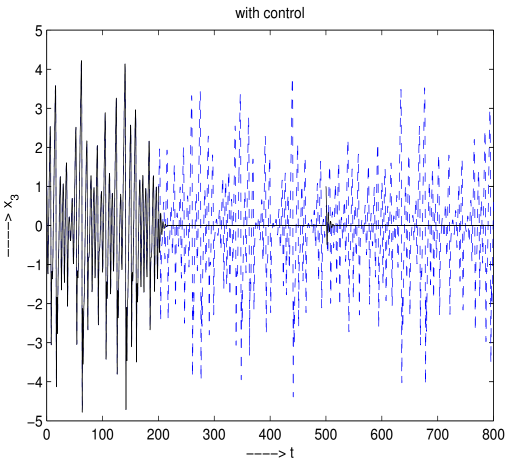

FIGURE 6. The time evolution of x_3 in controlled system

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