Bangmod Int. **J.** **M**ath. **&** **C**omp. **S**ci.
Vol. 1, No. 1, 2015: Pages 19 – 31
http://bangmod-jmcs.kmutt.ac.th/
ISSN : 2408-154X

EXISTENCE OF COINCIDENCE AND COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS IN SYMMETRIC SPACES

Saurabh Manro[1,*], **Anita Tomar**[2] **and Shin Min Kang**[3]
[1] *School of Mathematics and Computer Applications*
Thapar University
Patiala 147004, Punjab, INDIA
e-mail: sauravmanro@hotmail.com
[2] *V.S.K.C. Government Degree College,*
Dakpathar, Dehradun, India.
e-mail: anitatmr@yahoo.com
[3] *Department of Mathematics and RINS*
Gyeongsang National University
Jinju 660-701, KOREA
e-mail: smkang@gnu.ac.kr
**Corresponding author.*

Abstract The aim of this paper is to present a simple and unified approach for the existence of coincidence and common fixed point on symmetric spaces. Some illustrative examples is also furnished to demonstrate that our results are genuine improvement of many other known results existing in literature.

MSC: 47H10, 54H25

Keywords: Weak compatibility, symmetric space, common property $(E.A)$, CLR_S property, CLR_{ST} property, $JCLR_{ST}$ property, common fixed point.

Submission date: 7 April 2015 / Acceptance date: 28 April 2015 /Available online 29 April 2015

1. Introduction

Weak commutativity of a pair of mappings was introduced by Sessa [23] in fixed point considerations. There after number of generalizations of this notion have been obtained. Jungck [14] enlarged the class of noncommuting mappings by compatible mappings. Also the concept of compatible mappings was further improved by Jungck et al. [16] with the notion of weakly compatible mappings which merely commute at coincidence points. In fact weak compatibility is most widely used concept among all weaker forms of commuting mappings. For a brief development of weaker forms of commuting mappings one may

Published by **T**heoretical **a**nd **C**omputational **S**cience Center **(TaCS)**,
King Mongkut's University of Technology Thonburi **(KMUTT)**

refer to Singh et al. [25]. Recently fixed point theory for discontinuous and noncompatible mappings has attracted much attention. The property $(E.A)$, a generalization of noncompatibility, introduced by Aamri et al. [1] and common property $(E.A)$ introduced by Ali et al. [5] allows replacing the completeness requirement of the space with more natural condition of closedness of range and relaxes the continuity of mappings. Also the notion of common property $(E.A.)$ relaxes containment of range of one mapping into the range of other which is utilized to construct the sequence of joint iterates besides minimizing the commutativity conditions of the mappings to the commutativity at their points of coincidence.

On the other hand recently, Sintunavarat et al. [27] introduced the concept of the common limit in the range CLR property, Imdad et al. [17] Introduced the concept of CLR_{ST}, Manro et al. [18] introduced the concept of CLR_S property and Chauhan et al. [8] introduced the concept of $JCLR_{ST}$ property property which does not require even closedness of range for the existence of coincidence and common fixed point.

The aim of this paper is to present a simple and unified approach for the existence of coincidence and common fixed point in symmetric spaces.

2. Preliminaries

Recall that a *symmetric* on a nonempty set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$.

A set X together with a symmetric d is called a *symmetric space*. If d is symmetric on a set X, then for $x \in X$ and $\epsilon > 0$, we write $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is given by the sets $U \in \tau(d)$ (along with emptyset) in which for each $x \in U$, $B(x, \epsilon) \subset U$ for some $\epsilon > 0$. A symmetric d is a *semi-metric* if for each $x \in X$ and $\epsilon > 0$, $B(x, \epsilon)$ is a neighborhood of x in the topology $\tau(d)$.

A symmetric (respectively, semi-metric) space (X, d) is a topological space whose topology $\tau(d)$ on X is induced by symmetric (respectively, semi-metric) d.

Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$.

The distinction between symmetric and a semi-metric is evident as one can easily construct a symmetric d such that $B(x, \epsilon)$ need not be neighborhood of $x \in \tau(d)$.

The difference of a symmetric and a metric comes from the triangle inequality. Actually a symmetric space need not be Hausdroff. So in order to prove fixed point theorems some additional axioms are required. Wilson [28] gave the following axioms:

(W3) Given x_n, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$.

Aliouche [2] gave the following axioms:

(HE) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Pathak et al. [20] gave the following axioms:

(CE1) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

(CE2) Given $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X, $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\limsup_{n\to\infty} d(z_n, y_n) = \limsup_{n\to\infty} d(z_n, x_n)$.

Also, Cho et al. [9] gave the following axioms:

(CC) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

Note that if d is metric, then axioms (W3), (W4), (HE), (CE1) and (CE2) are automatically satisfied and if $\tau(d)$ is Hausdorff, then (W3) is satisfied. Cho et al. [9] carried out a systematic study of these crucial axioms via demonstrative examples which can be fruitful to the researchers of this domain. They proved that (W4) $\Rightarrow$ (W3) and (CC) or (CE1) $\Rightarrow$ (W3). Also (CC) implies all the remaining four conditions namely (W3), (W4), (HE) and (CE1). All other possible implications amongst (W3), (CE1) and (HE) are not true, in general.

A symmetric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X. It is interesting to note that in a symmetric space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if symmetric d is suitably restricted (see Wilson [28]). In 1972, Burke [7] furnished an illustrative example to show that a convergent sequence in a semi-metric space need not admit a Cauchy subsequence. Burke was able to formulate an equivalent condition under which every convergent sequence in a semi-metric space admits a Cauchy subsequence.

Recall, a subset S of a symmetrics space (X, d) is said to be *d-closed* if for a sequence $\{x_n\}$ in S and a point $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $x \in S$. For a symmetric space d-closedness implies $t(d)$ closedness and if d is a semi-metric, the converse is also true.

All the definitions in the metric space may be adapted in the setting of symmetric (semi-metric) space.

Definition 2.1. A pair of self mapping (S, T) on a symmetric space (X, d) is said to be

(1) *compatible* [14] if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

(2) *noncompatible* if there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sxn = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or nonexistent.

(3) *weakly compatible* [16] if $ASx = SAx$ for all $x \in X$ at which $Ax = Sx$.

Definition 2.2. A pair of self mapping (S, T) on a symmetric space (X, d) satisfies (1) *property* $(E.A)$ [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

(ii) *common limit in the range property* (CLR property) [27] if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Sz$ for some $z \in X$.

Clearly, both compatible and noncompatible pairs enjoy property $(E.A)$. Also notice that weak compatibility and property $(E.A)$ are independent of each other.

Definition 2.3. Two pairs of self mappings (A, S) and (B, T) is said to

(1) satisfy *common property* $(E.A)$ [5] if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$ for some $z \in X$.

(2) satisfy CLR_{ST} *property* (with respect to mappings S and T) [17] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X) \cap T(X)$.

(3) satisfy *$JCLR_{ST}$ property* (with respect to mappings S and T) [8] if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz = Tz$, where $z \in X$.

(4) *share common limit property in the range of S* (CLR_S property) [18] if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz$ for some $z \in X$.

Clearly common property $(E.A)$ contains property $(E.A)$.

If $A = B$ and $S = T$, then the above definition implies common limit in the range property (CLR property) due to Sintunavarat et al. [27]. Also notice that the preceding definition implies the common property $(E.A)$ but the reverse implication is not true in general.

Now, we give examples which points out difference between common property $(E.A)$, CLR_S property, CLR_{ST} property and $JCLR_{ST}$ property.

Example 2.4. Define A, B, S and T on $X = [1, 15)$ by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 14, & x \in (1, 3], \end{cases} \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 5, & x \in (1, 3], \end{cases}$$

$$Sx = \begin{cases} 3, & x = 1 \\ 6, & x \in (1, 3], \\ \frac{x+1}{4}, & x \in (3, 15), \end{cases} \quad Tx = \begin{cases} 2, & x = 1 \\ 11, & x \in (1, 3], \\ x - 2, & x \in (3, 15), \end{cases}$$

where $d(x, y) = (x - y)^2$.

Take $\{x_n\} = \{3 + \frac{1}{n}\}$ and $\{y_n\} = \{3 + \frac{1}{n}\}$. Clearly, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1$. Here

(1) $1 \in X$, so A, B, S and T satisfy common property $(E.A)$,

(2) $1 \neq S1$, so A, B, S and T do not satisfy CLR_S property,

(3) $1 \notin S(X) \cap T(X)$, so, A, B, S and T do not satisfy CLR_{ST} property,

(4) $1 \neq S1 \neq T1$, so A, B, S and T do not satisfy $JCLR_{ST}$ property,

(5) $A(X)$ and $B(X)$ are closed, but $S(X)$ and $T(X)$ are not closed.

Thus mappings satisfying common property $(E.A)$ need not satisfy CLR_S, CLR_{ST} and $JCLR_{ST}$ properties.

Example 2.5. Define A, B, S and T on $X = [1, 15)$ by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 14, & x \in (1, 3], \end{cases} \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 5, & x \in (1, 3], \end{cases}$$

$$Sx = \begin{cases} 1, & x = 1 \\ 6, & x \in (1, 3], \\ \frac{x+1}{4}, & x \in (3, 15), \end{cases} \quad Tx = \begin{cases} 2, & x = 1 \\ 11, & x \in (1, 3], \\ x - 2, & x \in (3, 15), \end{cases}$$

where $d(x, y) = (x - y)^2$.

Take $\{x_n\} = \{3 + \frac{1}{n}\}$ and $\{y_n\} = \{3 + \frac{1}{n}\}$. Clearly, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1$. Here

(1) $1 \in X$, so A, B, S and T satisfy common property $(E.A)$,

(2) $1 = S1$, so A, B, S and T satisfy CLR_S property,

(3) $1 \notin S(X) \cap T(X)$, so, A, B, S and T do not satisfy CLR_{ST} property,

(4) $1 = S1 \neq T1$, so A, B, S and T do not satisfy $JCLR_{ST}$ property,
(v) $A(X)$ and $B(X)$ are closed, but $S(X)$ and $T(X)$ are not closed.
Thus mappings satisfying CLR_S property satisfy common property $(E.A)$ but need not satisfy CLR_{ST} property and $JCLR_{ST}$ property.

Example 2.6. Define A, B, S and T on $X = [1, 15)$ by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 14, & x \in (1, 3], \end{cases} \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 5, & x \in (1, 3], \end{cases}$$

$$Sx = \begin{cases} 3, & x = 1 \\ 6, & x \in (1, 3], \\ \frac{x+1}{4}, & x \in (3, 15), \end{cases} \quad Tx = \begin{cases} 2, & x = 1 \\ 1, & x \in (1, 3], \\ x - 2, & x \in (3, 15), \end{cases}$$

where $d(x, y) = (x - y)^2$.

Take $\{x_n\} = \{3 + \frac{1}{n}\}$ and $\{y_n\} = \{3 + \frac{1}{n}\}$. Clearly, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1$. Here
(1) $1 \in X$, so A, B, S and T satisfy common property $(E.A)$,
(2) $1 = S1$, so A, B, S and T satisfy CLR_S property,
(3) $1 \in S(X) \cap T(X)$, so, A, B, S and T satisfy CLR_{ST} property,
(4) $1 = S1 \neq T1$, so A, B, S and T do not satisfy $JCLR_{ST}$ property,
(v) $A(X)$ and $B(X)$ are closed, but $S(X)$ and $T(X)$ are not closed.
Thus mappings satisfying CLR_{ST} property satisfy common property $(E.A)$ and CLR_S property but need not satisfy $JCLR_{ST}$ property.

Example 2.7. Define A, B, S and T on $X = [1, 15)$ by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 14, & x \in (1, 3], \end{cases} \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3, 15) \\ 5, & x \in (1, 3], \end{cases}$$

$$Sx = \begin{cases} 2, & x = 1 \\ 6, & x \in (1, 3], \\ \frac{x+1}{4}, & x \in (3, 15), \end{cases} \quad Tx = \begin{cases} 2, & x = 1 \\ 11, & x \in (1, 3], \\ x - 2, & x \in (3, 15), \end{cases}$$

where $d(x, y) = (x - y)^2$.

Take $\{x_n\} = \{3 + \frac{1}{n}\}$ and $\{y_n\} = \{3 + \frac{1}{n}\}$. Clearly, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1$. Here
(1) $1 \in X$, so A, B, S and T satisfies common property $(E.A)$,
(2) $1 = S1$, so A, B, S and T satisfy CLR_S property,
(3) $1 \in S(X) \cap T(X)$, so, A, B, S and T satisfy CLR_{ST} property,
(4) $1 = S1 = T1$, so A, B, S and T satisfy $JCLR_{ST}$ property,
(5) $A(X)$ and $B(X)$ are closed, but $S(X)$ and $T(X)$ are not closed.
Thus mappings satisfying $JCLR_{ST}$ property also satisfy common property $(E.A)$, CLR_S and CLR_{ST} properties.

Remark 2.8. In view of the preceding example notice that if two pairs of self mappings satisfy common property $(E.A)$ along with completeness/closedness of subspace, then it always enjoy CLR_S property, CLR_{ST} property and $JCLR_{ST}$ property.

Throughout this paper, $\mathbb{R}^+$ denotes the set of non-negative real numbers.

3. Implicit relation

In this paper we utilize integral type implicit relations due to their versatility of deducing several contraction conditions at the same time. Let $\mathbf{F}$ be the set of all continuous functions $F(t_1, t_2, \ldots, t_6) : \mathbb{R}^6 \to \mathbb{R}^+$ satisfying the following conditions:

$$(F_a) \qquad \int_0^{F(u,0,u,0,u,0)} \phi(t)dt \leq 0 \quad \text{implies} \quad u = 0;$$

$$(F_b) \qquad \int_0^{F(u,0,0,u,0,u)} \phi(t)dt \leq 0 \quad \text{implies} \quad u = 0$$

and the function $F(t_1, t_2, \ldots, t_6) : \mathbb{R}^6 \to \mathbb{R}^+$ satisfies the conditions (F_1) if

$$(F_1) \qquad \int_0^{F(u,u,0,0u,u)} \phi(t)dt > 0 \quad \text{for all } u > 0,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a summable Lebesgue-integrable map.

Example 3.1. Let $F(t_1, t_2, \ldots, t_6) = t_1 - c\min\{t_2, t_3, t_4, t_5, t_6\}$, where $0 < c < 1$ and $\phi(t) = t^2$ for all $t \in \mathbb{R}^+$. Then

$$\int_0^{F(u,0,u,0,u,0)} t^2 dt \leq 0, \quad \text{that is,} \int_0^u t^2 dt \leq 0,$$

which implies $u = 0$. Similarly,

$$\int_0^{F(u,0,0,u,0,u)} t^2 dt \leq 0, \quad \text{that is,} \int_0^u t^2 dt \leq 0,$$

which implies $u = 0$. So $F \in \mathbf{F}$ and the function F satisfies the conditions (F_a) and (F_b).

Further, function F satisfies the conditions (F_1) because

$$\int_0^{F(u,u,0,0,u,u)} t^2 dt > 0, \quad \text{that is,} \int_0^u t^2 dt = \frac{u^3}{3} > 0 \quad \text{for all } u > 0.$$

4. Main Results

Now we state our main results.

Theorem 4.1. *Let A, B, S and T be self mappings of a symmetric (semi-metric) space (X, d) which satisfy properties (HE) and $(W3)$. If*
$(C1)$ SX and TX are d-closed subspaces of X;

$$(C2) \qquad \int_0^{F(d(Ax,By),d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(Sx,By))} \phi(t)dt \leq 0$$

for all x,yI X, where $F \in \mathbf{F}$ and $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a summable Lebesgue-integrable map,
$(C3)$ (A, S) and (B, T) satisfy common property $(E.A)$.
Then pairs (A, S) and (B, T) have coincidence point. Further if (A, S) and (B, T) be weakly compatible pairs of self mappings of X, then A, B, S and T have a unique common fixed point in X.

Proof. In view of $(C3)$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$ for some $t \in X$. Since $S(X)$ is a d-closed subspace of X, therefore, there exists a point$u \in X$ such that $t = Su$.

We claim that $Au = t$. Using (HE) property and $(C2)$, take $x = u$ and $y = y_n$,

$$\int_0^{F(d(Au,By_n),d(Su,Ty_n),d(Au,Su),d(By_n,Ty_n),d(Au,Ty_n),d(Su,By_n))} \phi(t)dt \leq 0.$$

Taking limit as $n \to \infty$,

$$\int_0^{F(\lim_{n\to\infty} d(Au,By_n),0,\lim_{n\to\infty} d(Au,By_n),0,\lim_{n\to\infty} d(Au,Ty_n),0))} \phi(t)dt \leq 0,$$

which implies $\lim_{n\to\infty} d(Au, Bxn) = 0$ by using the condition (F_a). Thus, by $(W3)$ property, $Au = Su = t$, i.e., A and S have a coincidence point. Since $T(X)$ is also a d-closed subspace of X, therefore since $\lim_{n\to\infty} Ty_n = T$ in $T(X)$ and hence there exists $v \in X$ such that $Tv = t = Au = Su$. Now, by taking $x = u$ and $y = v$ in $(C2)$ we can easily show that $Bv = t$. Therefore, $Bv = t = Tv$, which shows that v is a coincidence point of the pair (B, T). Since the pair (A, S) is weakly compatible, it follow sthat $ASu = SAu$, that is,,$At = St$.

If $t \neq At$, using $(C2)$ we have

$$\int_0^{F(d(At,Bv),d(St,Tv),d(At,St),d(Bv,Tv),d(At,Tv),d(St,Bv))} \phi(t)dt \leq 0$$

and hence

$$\int_0^{F(d(At,t),d(At,t),0,0,d(At,t),d(At,t))} \phi(t)dt \leq 0$$

which is a contraction by (F_1). Hence $d(At, t) = 0$, that is, $t = At = St$. Thus. A and S have a common fixed point.

Similarly, the weak compatibility of B and T with $(C2)$ yields $t = Bt = Tt$, that is, B and T have a common fixed points. Thus t is a common fixed point of A, B, S and T.

For the uniqueness of common fixed point t, let w $(w \neq z)$ be another common fixed point of A, B, S and T. Then using $(C2)$, we have

$$\int_0^{F(d(At,Bw),d(St,Tw),d(At,St),d(Bw,Tw),d(At,Tw),d(St,Bw))} \phi(t)dt \leq 0$$

and hence

$$\int_0^{F(d(t,w),d(t,w),0,0,d(t,w),d(t,w))} \phi(t)dt \leq 0,$$

which is a contradiction by (F_1). Therefore $d(t, w) = 0$, that is, $t = w$. Hence, uniqueness follows. This completes the proof. ∎

Example 4.2. Let (X, d) be a symmetric space, where $X = [0, 2]$ and $d(x, y) = (x - y)^2$ satisfying the properties (HE) and $(W3)$. Define A, B, S and T by

$$Ax = Sx = \begin{cases} 0, & x \in [0,1), \\ 1, & x \in (1,2] \end{cases} \quad \text{and} \quad Bx = Tx = \begin{cases} 0, & x \in [0,1), \\ 2, & x \in (1,2]. \end{cases}$$

Take $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{1}{n}\}$. Clearly

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = 0 \in X.$$

Thus (A, S) and (B, T) satisfy common property $(E.A.)$. Also, $A(X) = S(X) = \{0, 1\}$ and $B(X) = T(X) = \{0, 2\}$. Clearly $A(X) \not\subset T(X)$ and $B(X) \not\subset S(X)$. Also the pairs (A, S) and (B, T) are weakly compatible and property $(CE2)$ is not satisfied. Moreover $S(X)$ and $T(X)$ are d-closed subspaces of X and condition $(C2)$ is satisfied by A, B, S and T. Thus, the mappings A, B, S and T satisfy all conditions of Theorem 4.1 and $x = 0$ is a unique common fixed point of A, B, S and T.

Remark 4.3. It is evident from the proof of the theorem that $t = Bv = Su$ and $t = Tv = Au$, that is, (A, S) and (B, T) may have different coincidences.

The following example supports this remark.

Example 4.4. Let $X = [0, 2]$ be equipped with symmetric $d(x, y) = (x-y)^2$. Let A, B, S and T be self mapping such that $Sx = 2x^2$, $Tx = 2x^4$, $Ax = x^2 + \frac{1}{4}$ and $Bx = x^4 + \frac{1}{4}$. Evidently, $A\frac{1}{2} = S\frac{1}{2}$ and $T2^{-\frac{1}{2}} = B^{-\frac{1}{2}}$, that is, S, A have a coincidence at $x = \frac{1}{2}$ and T, B have a (different) coincidence at $x = 2^{-\frac{1}{2}}$. Notice that $T2^{-\frac{1}{2}} = S\frac{1}{2}$.

Remark 4.5. It also improves the result due to Aamri and Moutawakil [1] if we define $F(t_1, t_2, t_3, t_4, t_5, t_6) - t_1 - c\max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$, where $0 < c < 1$ and $\phi(t) = 1$.

Moreover it is observed that common property $(E.A)$ require d-closedness of subspaces for the existence of fixed point, so attempt has been made to drop d-closedness of subspaces by using CLR_S property (CLR_T property).

Theorem 4.6. *Let A, B, S and T be self mappings of a symmetric (semi-metric) space (X, d) which satisfy properties (HE), $(W3)$ and condition $(C2)$ of Theorem 4.1. If*
$(C4)$ (A, S) and (B, T) shares the CLR_S property (CLR_T property);
$(C5)$ $A(X) \subset T(X)$ (or $B(X) \subset S(X)$).
Then pairs (A, S) and (B, T) have a coincidence point. Further if (A, S) and (B, T) be weakly compatible pairs of self mappings of X, then A, B, S and T have a unique common fixed point in X.

Proof. As the pairs (A, S) and (B, T) share the common limit in the range of S property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} T_n = \lim_{n\to\infty} By_n = Su$$

for some $u \in X$.

Firstly, we assert that $Au = Su$. By $(C2)$, we have

$$\int_0^{F(d(Au,By_n),d(Su,Ty_n),d(Au,Su),d(By_n,Ty_n),d(Au,Ty_n),d(Su,By_n))} \phi(t)dt \le 0.$$

Taking limit as $n \to \infty$, we have

$$\int_0^{F(\lim_{n\to\infty} d(Au,By_n),0,d(Au,By_n),0\lim_{n\to\infty} d(Au,Ty_n),0)} \phi(t)dt \le 0,$$

which implies $\lim_{n\to\infty} d(Au, By_n) = 0$ by using the condition (F_a). Thus by $(W3)$ property, $Au = Su = t$, that is, A and S have a coincidence point.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv$.

Secondly, we assert that $Bv = Tv$. By $(C2)$, we get

$$\int_0^{F(d(Au,Bv),d(Su,Tv),d(Au,Su),d(Bv,Tv),d(Au,Tv),d(Su,Bv))} \phi(t)dt \le 0$$

and hence

$$\int_0^{F(d(Tvu,Bv),0,0,d(Bv,Tv),0,d(Tv,Bv))} \phi(t)dt \leq 0,$$

which implies $d(Tv, Bv) = 0$ by using the condition (F_b). Thus, $Bv = Tv$ which shows that v is a coincidence point of the pair (B, T). Thus, we have $t = Tv = Bv = Au = Su$. The rest of proof is same as Theorem 4.1. ∎

Example 4.7. Let (X, d) be a symmetric space where $X = [1, 15)$ and $d(x, y) = (x-y)^2$ satisfying the properties $(H.E)$, $(W3)$ and condition $(C2)$. Define A, B, S and T by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3, 15), \\ x+6, & x \in (1, 3], \end{cases} \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3, 15), \\ x+5, & x \in (1, 3], \end{cases},$$

$$Sx = \begin{cases} 1, & x = 1, \\ 6x \in (1, 3], & \\ \frac{x+1}{4}, & x \in (3, 15), \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & x = 1, \\ 11x \in (1, 3], & \\ x-2, & x \in (3, 15). \end{cases}$$

Take $\{x_n\} = \{3+\frac{1}{n}\}$. Clearly $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = S1 = 1$. Thus, pairs (A, S) and (B, T) satisfy CLR_S property. Also, $A(X) = \{1\} \cup (7, 9]$, $B(X) = \{1\} \cup (6, 8]$, $S(X) = [1, 4) \cup \{6\}$ and $T(X) = [1, 13)$ so that $A(X) \subset T(X)$ but $B(X) \not\subset S(X)$. Also the pairs (A, S) and (B, T) are weakly compatible. Moreover it should be noted that AX, BX, SX and TX are not d-closed subspaces of X and property $(CE2)$ is not satisfied. Thus the mappings A, B, S and T satisfy all conditions of Theorem 4.6 and $x = 1$ is a common fixed point of mappings A, B, S and T.

Now we attempt to drop containment of range of one mapping into the range of other from above theorem by replacing CLR_S property by $JCLR_{ST}$ property

Theorem 4.8. *Let A, B, S and T be four self mappings in symmetric metric space which satisfy properties (HE), $(W3)$ and condition $(C2)$ of Theorem 4.1. If (A, S) and (B, T) satisfy $JCLR_{ST}$ property, then pairs (A, S) and (B, T) have coincidence point. Further if (A, S) and (B, T) be weakly compatible pairs of self mappings of X, then A, B, S and T have a unique common fixed point in X.*

Proof. As the pairs (A, S) and (B, T) satisfy the $JCLR_{ST}$ property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = Sz = Tz$ for some $z \in X$.

Firstly, we assert that $Az = Sz$. By $(C2)$, we have

$$\int_0^{F(d(Az,By_n),d(Sz,Ty_n),d(Az,Sz),d(By_n,Ty_n),d(Az,Ty_n),d(Sz,By_n))} \phi(t)dt \leq 0.$$

Taking limit as $n \to \infty$, we have

$$\int_0^{F(\lim_{n\to\infty} d(Az,By_n),0,(\lim_{n\to\infty} d(Az,By_n)0,(\lim_{n\to\infty} d(Az,By_n),0)} \phi(t)dt \leq 0,$$

which implies $\lim_{n\to\infty} d(Az, By_n) = 0$ by using the condition (F_a). Thus, by $(W3)$ property, which implies $d(Az, Sz) = 0$, by using the condition (F_a) which shows that $Az = Sz$, that is z is a coincidence point of the pair (A, S).

Secondly, we assert that $Bz = Tz$. By $(C2)$, we get

$$\int_0^{F(d(Az,Bz),d(Sz,Tz),d(Az,Sz),d(Bz,Tz),d(Az,Tz),d(Sz,Bz))} \phi(t)dt \leq 0$$

and hence

$$\int_0^{F(d(Az,Bz),0,0,d(Bz,Tz),0,d(Tz,Bz))} \phi(t)dt \leq 0,$$

which implies $d(Tz, Bz) = 0$, by using the condition (F_b) which show sthat $Tz = Bz$, that is z is a coincidence point of the pair (B, T). Thus, we have $Tz = Bz = Az = Sz$. The rest of proof is same as Theorem 3.1. ∎

In the following theorem we attempt drop containment of range of one mapping into the range of other and d-closedness of subspaces by using CLR_{ST} property.

Theorem 4.9. *Let A, B, S and T be four self mappings in symmetric space (X, d) satisfying properties (HE), $(W3)$ and condition $(C2)$. If pairs (A, S) and (B, T) satisfy CLR_{ST} property, then pairs (A, S) and (B, T) have coincidence point. Further if (A, S) and (B, T) be weakly compatible pairs of self mappings of X, then A, B, S and T have a unique common fixed point in X.*

Proof. As the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, that is, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X) \cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$.

Firstly, we assert that $Au = Su$. By $(C2)$, we have

$$\int_0^{F(d(Au,By_n),d(Su,Ty_n),d(Au,Su),d(By_n,Ty_n),d(Au,Ty_n),d(Su,By_n))} \phi(t)dt \leq 0.$$

Taking limit as $n \to \infty$, we get

$$\int_0^{F(\lim_{n\to\infty} d(Au,By_n),0,\lim_{n\to\infty} d(Au,Sy_n),0,\lim_{n\to\infty} d(Au,Ty_n),0)} \phi(t)dt \leq 0,$$

which implies $\lim_{n\to\infty} d(Au, Byn) = 0$, by using the condition (F_a). Thus, by $(W3)$ property, $d(Au, Su) = 0$, by using the condition (F_a) This gives $Au = Su = z$ which shows that u is a coincidence point of the pair (A, S).

Also, as $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$.

Secondly, we can easily prove $Bv = Tv$ by taking $x = u$ and $y = v$ in $(C2)$, we get $Bv = Tv = z$ which shows that v is a coincidence point of the pair (B, T). Thus, we have $z = Tv = Bv = Au = Su$. The rest of proof is same as Theorem 4.1. ∎

Example 4.10. Let (X, d) be a symmetric space where $X = [1, 15)$ and $d(x, y) = (x-y)^2$ satisfying the properties (HE), $(W3)$ and the condition $(C2)$. Define A, B, S and T on X by

$$Ax = \begin{cases} 1, & x \in \{1\} \cup (3, 15), \\ x + 11, & x \in (1, 3], \end{cases} \quad Bx = \begin{cases} 1, & x \in \{1\} \cup (3, 15), \\ x + 5, & x \in (1, 3], \end{cases},$$

$$Sx = \begin{cases} 1, & x = 1, \\ 6x \in (1, 3], & \\ \frac{x+1}{4}, & x \in (3, 15), \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & x = 1, \\ 11x \in (1, 3], & \\ x - 2, & x \in (3, 15). \end{cases}$$

Take $\{x_n\} = \{3+\frac{1}{n}\}$. Clearly $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = 1 \in S1 \cap T1$. Hence pairs (A, S) and (B, T) satisfy CLR_{ST} property and are weakly compatible. Moreover it should be noted that $A(X)$, $B(X)$, $S(X)$ and $T(X)$ are not d-closed subspaces of X. Also $A(X) \not\subset T(X)$ and $B(X) \not\subset S(X)$ and the property $(CE2)$ is not satisfied. Thus the mappings A, B, S and T satisfy all conditions of Theorem 4.9 and $x = 1$ is the common fixed point of self mappings A, B, S and T.

Remark 4.11. Every contractive condition of integral type automatically includes a corresponding contractive condition, not involving integrals by setting $\phi(t) = 1$ so corresponding results of the form are also generalized and improved. In particular our results improve results of Popa [21], [22], Imdad et al. [11], Pathak et al. [20], Aamri et al. [1] and Cho et al. [9]. It also improves the result of Aliouche et al. [4] using the fact that every metric space is a symmetric space.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (1) (2002), 181–188.

[2] A. Aliouche, *A common fixed point theorem for weakly compatible mapping in symmetric spaces satisfying a contractive condition of integral type*, J. Math. Anal. Appl., **322** (2) (2006), 796–802.

[3] A. Aliouche and A. Djoudi, *A general common fixed point theorem for reciprocally continuous mappings satisfying an implicit relation*, Aust. J. Math. Anal. Appl., **3** (1) (2006), Article 9, 7 pages.

[4] A. Aliouche and A. Djoudi, *Common fixed point theorems for satisfying an implicit relation without decreasing assumption*, Hacet. J. Math. Stat., **36** (1) (2007), 11–18.

[5] J. Ali, M. Imdad and D. Bahuguna, *Common fixed point theorems in Menger spaces with common property (E.A)*, Comput. Math. Appl., **60**(12) (2010) 3152–3159.

[6] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29** (9) (2002), 531–536.

[7] D. K. Burke, *Cauchy sequences in semimetric spaces*, Proc. Amer. Math. Soc, **33** (1972), 161–165.

[8] S. Chauhan, W. Sintunavarat and P. Kumam, *Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using (JCLR) property*, Appl. Math., **3**(9) (2012), 976–982.

[9] S.H. Cho, G.Y. Lee and J.S. Bae, *On coincidence and fixed-point theorems in symmetric spaces*, Fixed Point Theory Appl., 2008 (2008), Article ID 562130, 9 pages.

[10] T.L. Hicks and B.E. Rhoades, *Fixed point theory with applications to probabilistic spaces*, Nonlinear Anal., **36** (3) (1999), 331–344.

[11] M. Imdad, S. Kumar and M.S. Khan, *Remarks on fixed point theorems satisfying implicit relations*, Rad. Mat., **11** (1) (2002), 135–143.

[12] M. Imdad and J. Ali, Jungcks common fixed point theorem and E.A property, Acta Math. Sin. (Engl. Ser.), **24** (1) (2008), 87–94.

[13] J. Jachymski, Common fixed point theorems for some families of maps, Indian. J. Pure Appl. Math., **25** (1994), 925–937.

[14] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., **9** (4) (1986), 771–779.
[15] G. Jungck, K. B. Moon, S. Park and B.E. Rhoades, *On generalization of the Meir-Keeler type contraction maps: corrections*, J. Math. Anal. Appl., **180** (1) (1993), 221–222.
[16] G. Jungck and B.E. Rhoades, *Fixed Point for set valued functions without continuity*, Indian J. Pure Appl. Math, **29** (3) (1998), 227–238.
[17] M. Imdad, B. D. Pant and S. Chauhan, *Fixed point theorems in Menger spaces using the CLR_{ST} property and applications*, J. Nonlinear Anal. Optim., **3** (2) (2012), 225–237.
[18] S. Manro and C. Vetro, *Common fixed point theorems in fuzzy metric spaces employing CLRS and JCLRST properties*, Facta Universitatis-Ser. Math. Inform., **29** (2014), 77–90.
[19] R.P. Pant, *Common fixed points of sequences of mappings*, Ganita, **47** (2) (1996), 43–49.
[20] H.K. Pathak, R. Tiwari and M.S. Khan, *A common fixed point theorem satisfying integral type implicit relations*, Appl. Math E-Notes, **7** (2007), 222–228.
[21] V. Popa, *Some fixed point theorems for compatible mapping satisfying an implicit relation*, Demonstratio Math., **32** (1) (1999), 157–163.
[22] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet Ştiinţ. Ser. Mat. Univ. Bacău , **7** (1997), 127–133.
[23] S. Sessa, *On a weak commutativity condition in fixed point consideration*, Publ. Inst. Math. (Beograd), **32** (46) (1982), 133–146.
[24] S.L. Singh, *Coincidence theorems and convergence of the sequences of coincidence values*, Punjab Univ. J. Math. (Lahore) **19** (1986), 83–97.
[25] S.L. Singh and A. Tomar, *Weaker forms of community maps and existence of fixed points*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., **10** (3) (2003), 145–161.
[26] S.L. Singh and B. D. Pant, *Coincidence and fixed point theorems for a family of maps on Menger spaces and extension to uniform spaces*, Math. Japon., **33** (6) (1988), 957–963.
[27] W. Sintunavarat and P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math, 2011 (2011), Article ID 637958, 14 pages.
[28] W.A. Wilson, *On semi-metric spaces*, Amer. J. Math., **53** (1931), 361–373.

Bangmod International
Journal of **M**athematical **C**omputational **S**cience
ISSN: 2408-154X
Bangmod-JMCS Online @ http://bangmod-jmcs.kmutt.ac.th/

Journal office:
Theoretical and Computational Science Center (TaCS)

Science Laboratory Building, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, Thailand 10140
Website: http://tacs.kmutt.ac.th/
Email: tacs@kmutt.ac.th