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STRONG CONVERGENCE THEOREMS FOR TOTAL ASYMPTOTICALLY STRICT QUASI-ϕ PSEUDOCONTRACTIVE NONSELF MAPPINGS IN BANACH SPACES

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Abstract The purpose of this paper is to introduce the notion of total asymptotically strict quasi-ϕ pseudocontractive nonself mappings. By using hybrid projection algorithms and under suitable conditions some strong convergence theorems for a total asymptotically strict quasi-ϕ pseudocontractive nonself mapping are obtained in the setting of uniformly convex and smooth Banach spaces. The results presented in this paper extend and improve the corresponding results of Qin et al [Fixed Point Theory and Application 143(2012)], Qin et al [Abstract and Applied Analysis, (2011), ID 142626], Zhang [Fixed Point Theory and Applications, 137(2012)], Chang et al [Applied Mathematics and Computation, 218(11),(2012), 6489-6497], and many other authors.

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1. Introduction and preliminaries

Throughout this article we assume that E is a real Banach space with norm $\|\cdot\|$, E^* is the dual space of E, $\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^*, C is a nonempty closed convex subset of E, $U = \{x \in E : \|x\| = 1\}$ is the unit sphere of E, $\mathbb{N}$ and $\mathbb{R}^+$

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are the set of the natural numbers and the set of nonnegative real numbers, respectively. $J : E \to 2^{E^*}$ is *the normalized duality mapping* defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2; \|f^*\| = \|x\|, x \in E.$$

Let $T : C \to C$ be a nonlinear mapping, $F(T)$ denote the set of fixed points of mapping T.

A subset C of E is said to be *retract of E* if there exists a continuous mapping $P : E \to C$ such that $Px = x$ for all $x \in C$. A mapping $P : E \to C$ is said to be a *retraction* if $P^2 = P$. Note that if P is a retraction, then $Pz = z$ for all $z \in R(P)$, the range of P. A Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for $x,\ y \in U$, $x \neq y$. X is said to be *uniformly convex* if for every $\epsilon \in (0, 2]$, there is $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1-\delta$ for any $x,\ y \in U$ with $\|x - y\| \geq \epsilon$. Every nonempty closed convex subset of a uniformly convex Banach space is a retraction. A Banach space E is said to be *smooth*, if $lim_{t\to 0} \frac{\|x+ty\|-\|y\|}{t}$ exists for each $x, y \in U$. X is said to be *uniformly smooth*, if the above limit is attained uniformly for each $x, y \in U$. It is well known that if E is reflexive and smooth, then J is surjective and single valued. A Banach space E is said to have the *Kadec-Klee property*, if for any sequence $\{x_n\}$ of E with $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

In the sequel, we assume that E is a smooth, uniformly convex and reflexive Banach space and C is a nonempty closed convex subset of E. We use $\phi : E \times E \to \mathbb{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \forall x, y \in E. \tag{1.1}$$

It is obvious that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \forall x, y \in E \tag{1.2}$$

and

$$\begin{aligned} &\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \leq \lambda\phi(x, y) + (1-\lambda)\phi(x, z), \\ &\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \forall x, y, z \in E. \end{aligned} \tag{1.3}$$

The generalized projection [1] $\Pi_C x : E \to C$ is defined by

$$\Pi_C x = \arg\inf_{y\in C} \phi(y, x), \forall x \in E.$$

Lemma 1.1. *[1] Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*
(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C, y \in E$;
(ii) If $x \in E$ and $z \in C$, then $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0, \forall y \in C$;
(iii) For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Lemma 1.2. *[2] Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Recently, many researchers have focused on studying the convergence of iterative scheme for quasi-ϕ-asymptotically nonexpansive mappings (See [3–6]) and total quasi-ϕ-asymptotically nonexpansive mappings (see [7–10]). The class of strict quasi-ϕ- pseudo-contractions was first considered by Zhou et al[11]. Qin et al [12] first considered the class of asymptotically strict quasi-ϕ-pseudocontractions. Zhang [13] established some strong convergence theorems of fixed points for asymptotically strict quasi-ϕ-pseudocontractions by hybrid projection algorithms. Qin et al [14] introduced the notion of asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense, and proved some strong convergence theorems to a fixed point in a real Banach space.

Definition 1.3. (1)[11] A mapping $T : C \to C$ is said to be strict quasi-ϕ pseudocontractive, if $F(T) \neq \emptyset$ and there exists a constant $\kappa \in [0, 1)$ such that

$$\phi(p, Tx) \leq \phi(p, x) + \kappa\phi(x, Tx), \forall x \in C, p \in F(T).$$

(2)[12] A mapping $T : C \to C$ is said to be asymptotically strict quasi-ϕ pseudocontractive, if $F(T) \neq \emptyset$ and there exist a sequences $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$, such that

$$\phi(p, T^n x) \leq k_n\phi(p, x) + \kappa\phi(x, T^n x), \forall x \in C, p \in F(T), n \in \mathbb{N}.$$

(3)[9] A mapping $T : C \to C$ is said to be total asymptotically quasi-ϕ nonexpansive mapping, if $F(T) \neq \emptyset$ and there exist sequences $\{\mu_n\}, \{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \mu_n\psi(\phi(p, x)) + \nu_n$$

holds for all $x \in C, p \in F(T)$ and all $n \in \mathbb{N}$.

Definition 1.4. A mapping $T : C \to C$ is said to be total asymptotically strict quasi-ϕ pseudocontractive, if $F(T) \neq \emptyset$, and there exist sequences $\{\mu_n\}, \{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ and a constant $\kappa \in [0, 1)$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \mu_n\psi(\phi(p, x)) + \nu_n + \kappa\phi(x, T^n x), \forall x \in C, p \in F(T), n \in \mathbb{N}. \quad (1.4)$$

Remark 1.5. Obviously the class of total asymptotically strict quasi-ϕ pseudocontractive mappings include asymptotically strict quasi-ϕ pseudocontractive mappings and total asymptotically quasi-ϕ nonexpansive mappings as special cases.

Now, we give some examples of total asymptotically strict quasi-ϕ pseudocontractive mapping.

Example 1.6. Let B be a unit ball in a real Hilbert $l^2 := \{x = (x_1, x_2, \cdots) | \sum_{i=1}^{\infty} |x_n|^2 \leq \infty\}$, and let $T : B \to B$ be a mapping defined by

$$T : (x_1, x_2, \cdots,) \to (0, x_1^2, a_2x_2, a_3x_3, \cdots)$$

where $\{a_i\}$ is a sequence in $(0, 1)$ such that $(2\prod_{i=2}^{n} a_j) - 1 > 0$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. Then

$$\begin{aligned}\phi(p, T^n x) &= \|p - T^n x\|^2 \\ &\leq 2(\prod_{i=2}^{n} a_i)\|p - Tx\|^2 + \kappa\|x - T^n x\|^2 \\ &= 2(\prod_{i=2}^{n} a_i)\phi(p, Tx) + \kappa\phi(x, T^n x), \forall x \in C, n \geq 2. \quad (1.5)\end{aligned}$$

Denote by $\mu_1 = 2, \mu_n = (2\prod_{i=2}^{n} a_j) - 1$, $n \geq 2$, $\psi(\phi(p, x)) = \phi(p, x)$, then we have

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} (2\prod_{i=2}^{n} a_j)^2 = 1.$$

Letting $\kappa = \nu_n = 0$, then $\forall x, y \in C,\ n \geq 1$, we have

$$\phi(p, T^n x) \leq \phi(p, x) + \mu_n \psi(\phi(p, x)) + \nu_n + \kappa\phi(x, T^n x), \forall x \in B, p \in F(T), n \in \mathbb{N}.$$

This implies that T is a total asymptotically strict quasi-ϕ pseudocontraction mapping.

Example 1.7. Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2},\ \forall x = (x_1, x_2, \cdots, x_n, \cdots) \in X,$$

and $C = \{x = (x_1, x_2, \cdots, x_n, \cdots) | x_i \in R^1, i = 1, 2, \cdots\}$ be an orthogonal subspace of X (i.e., $\forall x, y \in C$, we have $\langle x, y\rangle = 0$). It is obvious that C is a nonempty closed convex subset of X. For each $x = (x_1, x_2, \cdots, x_n, \cdots) \in C$, we define a mapping $T : C \to C$ by

$$Tx = \begin{cases} (x_1, x_2, \cdots, x_n, \cdots),\ if\ \Pi_{i=1}^{\infty} x_i < 0; \\ (-x_1, -x_2, \cdots, -x_n, \cdots),\ if\ \Pi_{i=1}^{\infty} x_i \geq 0. \end{cases} \tag{1.6}$$

Next we prove that T is a total asymptotically strict quasi-ϕ pseudocontractive mapping.

In fact, for any $x \in C, p \in F(T)$, denote by $\psi(\phi(p, x)) = \phi(p, x)$. Now we consider the following two cases.

Case 1. If $\Pi_{i=1}^{\infty} x_i < 0$, then we have $T^n x = x$ and so

$$\begin{aligned} \phi(p, T^n x) &= \|p - T^n x\|^2 = \|p - x\|^2 \\ &\leq \phi(p, x) + \mu_n \psi(\phi(p, x)) + \nu_n + \kappa\phi(x, T^n x), n \in \mathbb{N}, \end{aligned}$$

then inequality (1.4) holds.

Case 2. If $\Pi_{i=1}^{\infty} x_i \geq 0$, then we have $T^n x = (-1)^n x$. Hence we have

$$\begin{aligned} \phi(p, T^n x) &= \|p - T^n x\|^2 = \|p - (-1)^n x\|^2 = \|p\|^2 + \|x\|^2 \\ &\leq \phi(p, x) + \mu_n \psi(\phi(p, x)) + \nu_n + \kappa\phi(x, T^n x), n \in \mathbb{N}, \end{aligned}$$

thus the inequality (1.4) still holds. Therefore the mapping defined by (1.6) is a total asymptotically strict quasi-ϕ pseudocontractive mapping.

Example 1.8. [15] Let E be a uniformly smooth and strictly convex Banach space and $A : E \to E^*$ be a maximal monotone mapping such that $A^{-1}0 \neq \emptyset$, then $J_r = (J + rA)^{-1} J$ is a closed and quasi-ϕ-nonexpansive mapping from E onto $D(A)$, and so it is a total asymptotically strict quasi-ϕ pseudocontractive mapping.

Example 1.9. [3] Let Π_C be the generalized projection from a smooth, reflexive and strictly convex Banach space E onto a nonempty closed convex subset C of E, then Π_C is a closed and quasi-ϕ-nonexpansive from E onto C. Therefore, it is a total asymptotically strict quasi-ϕ pseudocontractive mapping.

Recently the strong and weak convergence of nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see for example [16–23]). Especially Chang [16] studied the convergence theorems for a countable family of quasi-ϕ-asymptotically nonexpansive nonself mappings in the setting of Banach spaces by using the modified Halpern and Mann-type iteration algorithm. Now we give the following definitions.

Definition 1.10. Let $P : E \to C$ be a retraction.

(1) [16] A mapping $T : C \to E$ is said to be quasi-ϕ asymptotically nonexpansive nonself mapping, if $F(T) \neq \emptyset$, and there exist a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ such that

$$\phi(u, T(PT)^{n-1}x) \leq k_n\phi(u, x)\ \forall x \in C,\ u \in F(T),\ and\ n \geq 1.$$

(2) A mapping $T : C \to E$ is said to be asymptotically strict quasi-ϕ pseudocontractive nonself mapping, if $F(T) \neq \emptyset$, and there exist a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ and a constant $\kappa \in [0, 1)$ such that for all $x \in C, u \in F(T)$ and all $n \geq 1$

$$\phi(u, T(PT)^{n-1}x) \leq k_n\phi(u, x) + \kappa\phi(x, T(PT)^{n-1}x)$$

.

(3) A mapping $T : C \to E$ is said to be total asymptotically strict quasi-ϕ pseudocontractive nonself mapping, if $F(T) \neq \emptyset$ and there exist sequences $\{\mu_n\}, \{\nu_n\}$ with $\mu_n, \nu_n \to 0$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ and a constant $\kappa \in [0, 1)$ such that for all $x \in C, u \in F(T)$ and all $n \geq 1$

$$\phi(u, T(PT)^{n-1}x) \leq \phi(u, x) + \mu_n\psi(\phi(u, x)) + \nu_n + \kappa\phi(x, T(PT)^{n-1}x)$$

. (4) A nonself mappings $T : C \to E$ is said to be asymptotically regular on C, if for any bounded subset K of C, the following holds

$$\limsup_{n\to\infty, x\in K} \{\|T(PT)^n x - T(PT)^{n-1}x\|\} = 0$$

The purpose of this paper is by using hybrid projection algorithms to prove some strong convergence to a fixed point of total asymptotically strict quasi-ϕ pseudocontractive nonself mappings in the framework of Banach spaces. The results presented in this article improve and extend the corresponding results of [7–14] and many others.

In order to prove our main results, we need the following lemma.

Lemma 1.11. *Let E be a real uniformly convex and smooth Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be a continuous total asymptotically strict quasi-ϕ pseudocontractive nonself mapping with respect to P defined by Definition 1.10, if $\nu_1 = 0$, then the fixed point set $F(T)$ is a closed and convex set of C.*

Proof. Let $\{u_n\}$ be a sequence in $F(T)$ such that $u_n \to u (n \to \infty)$, we need to prove that $u \in F(T)$.

Since $T : C \to E$ is a continuous total asymptotically strict quasi-ϕ pseudocontractive nonself mapping, we have

$$\phi(u, Tu) = \lim_{n\to\infty} \phi(u_n, Tu) \leq \lim_{n\to\infty} [\phi(u_n, u) + \mu_1\psi(\phi(u_n, u)) + \nu_1 + \kappa\phi(u, Tu)]. \quad (1.7)$$

This together with $\nu_1 = 0$, we have

$$\phi(u, Tu) \leq \frac{1}{1-\kappa} \lim_{n\to\infty} [\phi(u_n, u) + \mu_1\psi(\phi(u_n, u)) + \nu_1] = 0. \quad (1.8)$$

By Lemma 1.1(iii) and (1.8), we have $u = Tu$. This implies that $F(T)$ is closed.

We now prove that $F(T)$ is convex. Let $u_1, u_2 \in F(T)$ and $u = tu_1 + (1 - t)u_2$, where $t \in (0, 1)$. From the definition of T, we see that

$$\phi(u_1, T(PT)^{n-1}u) \leq \phi(u_1, u) + \mu_n\psi(\phi(u_1, u)) + \nu_n + \kappa\phi(u, T(PT)^{n-1}u)$$

and

$$\phi(u_2, T(PT)^{n-1}u) \leq \phi(u_2, u) + \mu_n\psi(\phi(u_2, u)) + \nu_n + \kappa\phi(u, T(PT)^{n-1}u).$$

On the other hand, we obtain from (1.3) that

$$\phi(u_1, T(PT)^{n-1}u) = \phi(u_1, u) + \phi(u, T(PT)^{n-1}u) + 2\langle u_1 - u, Ju - JT(PT)^{n-1}u\rangle$$

and

$$\phi(u_2, T(PT)^{n-1}u) = \phi(u_2, u) + \phi(u, T(PT)^{n-1}u) + 2\langle u_2 - u, Ju - JT(PT)^{n-1}u\rangle.$$

So we can deduce from above that

$$\phi(u, T(PT)^{n-1}u) \leq \frac{2\langle u - u_1, Ju - JT(PT)^{n-1}u\rangle + \mu_n\psi(\phi(u_1, u)) + \nu_n}{1-\kappa} \tag{1.9}$$

and

$$\phi(u, T(PT)^{n-1}u) \leq \frac{2\langle u - u_2, Ju - JT(PT)^{n-1}u\rangle + \mu_n\psi(\phi(u_2, u)) + \nu_n}{1-\kappa}. \tag{1.10}$$

Multiplying t and $1-t$ on the both sides of (1.9) and (1.10), respectively, it yields that

$$\phi(u, T(PT)^{n-1}u) \leq \frac{\mu_n[t\psi(\phi(u_1, u)) + (1-t)\psi(\phi(u_2, u))] + \nu_n}{1-\kappa}. \tag{1.11}$$

Hence we have

$$\lim_{n\to\infty} \phi(u, T(PT)^{n-1}u) = 0.$$

In light of *Lemma* 1.2, we obtain that

$$\lim_{n\to\infty} \|T(PT)^{n-1}u\| = \|u\| \text{ and } \lim_{n\to\infty} \|J(T(PT)^{n-1}u)\| = \|Ju\|. \tag{1.12}$$

Since E^* is reflexive, without loss of generality, we may assume that $J(T(PT)^{n-1}u) \rightharpoonup e^* \in E^*$. In view of the reflexivity of E, we have $JE = E^*$. So there exists an element $e \in E$, such that $Je = e^*$. It follows from (1.1) that

$$\begin{aligned}\phi(u, T(PT)^{n-1}u) &= \|u\|^2 - 2\langle u, J(T(PT)^{n-1}u)\rangle + \|T(PT)^{n-1}u\|^2\\ &= \|u\|^2 - 2\langle u, J(T(PT)^{n-1}u)\rangle + \|J(T(PT)^{n-1}u)\|^2.\end{aligned}$$

Taking $\lim_{n\to\infty}$ on the both sides of the above equality, we obtain that

$$\begin{aligned}0 = \|u\|^2 - 2\langle u, e^*\rangle + \|e^*\|^2 &= \|u\|^2 - 2\langle u, Je\rangle + \|Je\|^2\\ &= \|u\|^2 - 2\langle u, Je\rangle + \|e\|^2 = \phi(u, e)\end{aligned} \tag{1.13}$$

which implies that $u = e$, that is $Ju = e^*$, so that $J(T(PT)^{n-1}u) \rightharpoonup Ju \in E^*$. By the Kadec-Klee property of E^*, we obtain from (1.12) that

$$\lim_{n\to\infty} \|J(T(PT)^{n-1}u) - Ju\| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T(PT)^{n-1}u \rightharpoonup u$. By virtue of the Kadec-Klee property of E, we see from (1.12) that $T(PT)^{n-1}u \to u$ as $n \to \infty$. Hence $T(PT)^n u \to u$ as $n \to \infty$, i.e. $TP[T(PT)^{n-1}u] \to u$ as $n \to \infty$. In view of the continuity of TP, we can obtain that $TPu = u$. Since $u \in C, Pu = u$, we get $Tu = u$. So $F(T)$ is convex. The proof of Lemma 1.11 is completed.

2. Main Results

Theorem 2.1. *Let E be a real uniformly convex and smooth Banach space, C be a nonempty closed convex subset of E. Let $T : C \to E$ be a continuous and total asymptotically strict quasi-ϕ pseudocontractive nonself mapping. Suppose T is asymptotically regular and $F(T)$ is nonempty and bounded. Suppose there exist $M^* > 0$, such that $\psi(\eta) \leq M^*\eta$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E, \text{chosen arbitrarily; } C_1 = C \\ C_{n+1} = \{z \in C_n : \phi(x_n, T(PT)^{n-1}x_n) \leq \dfrac{2}{1-\kappa}\langle x_n - z, Jx_n - JT(PT)^{n-1}x_n\rangle + \xi_n\} \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \forall n \geq 1. \end{cases} \tag{2.1}$$

where $\xi_n = \frac{\mu_n M^* \sup_{p\in F(T)} \phi(p,x_n)+\nu_n}{1-\kappa}$. If $\nu_1 = 0$, then the iterative sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$ which is a fixed point of T in C.

Proof. **(I)** We prove that $F(T)$ and $C_n (n \in \mathbb{N})$ are all closed and convex subsets in C.

Indeed it follows from Lemma 1.11 that $F(T)$ is a closed and convex subset of C. $C_n(n \in \mathbb{N})$ is obviously closed. By the assumption we know that $C_1 = C$ is convex. We suppose that C_n is convex for some $n \geq 2$. We now show C_{n+1} is convex. Let $z_1, z_2 \in C_{n+1}$, we have that

$$\phi(x_n, T(PT)^{n-1}x_n) \leq \frac{2}{1-\kappa}\langle x_n - z_1, Jx_n - JT_i(PT_i)^{n-1}x_n\rangle + \xi_n$$

and

$$\phi(x_n, T(PT)^{n-1}x_n) \leq \frac{2}{1-\kappa}\langle x_n - z_2, Jx_n - JT(PT)^{n-1}x_n\rangle + \xi_n.$$

Let $z = tz_1 + (1-t)z_2$, where $t \in (0,1)$, we can get that

$$\phi(x_n, T(PT)^{n-1}x_n) \leq \frac{2}{1-\kappa}\langle x_n - z, Jx_n - JT(PT)^{n-1}x_n\rangle + \xi_n.$$

This shows that C_{n+1} is convex.

(II) We prove that $F(T) \subset C_n, \forall n \in \mathbb{N}$.

In fact $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_n$ for some $n \geq 2$. For any $u \in F(T) \subset C_n$, it follows from (1.3) that

$$\phi(u, T(PT)^{n-1}x_n) = \phi(u, x_n) + \phi(x_n, T(PT)^{n-1}x_n) + 2\langle u - x_n, Jx_n - JT(PT)^{n-1}x_n\rangle. \tag{2.2}$$

From equality (2.2) and

$$\phi(u, T(PT)^{n-1}x_n) \leq \phi(u, x_n) + \mu_n\psi(\phi(u, x_n)) + \nu_n + \kappa\phi(x_n, T(PT)^{n-1}x_n),$$

we obtain that

$$\phi(x_n, T(PT)^{n-1}x_n) \leq \frac{2}{1-\kappa}\langle x_n - u, Jx_n - JT(PT)^{n-1}x_n\rangle + \xi_n \tag{2.3}$$

where $\xi_n = \frac{\mu_n M^* \sup_{p\in F(T)} \phi(p,x_n)+\nu_n}{1-\kappa}$. This shows that $u \in C_{n+1}$, so $F(T) \subset C_{n+1}$.

(III) We prove that $\{x_n\}$ is a convergent sequence in C.

Since $x_n = \prod_{C_n} x_1$, from Lemma 1.1(ii) we have

$$\langle x_n - y, Jx_1 - Jx_n\rangle \geq 0, \forall y \in C_n.$$

Again since $F(T) \subset C_n, n \geq 1$, we have

$$\langle x_n - u, Jx_1 - Jx_n\rangle \geq 0, \forall u \in F(T). \tag{2.4}$$

It follows from Lemma 1.1(i) that for each $u \in F(T)$, $n \geq 1$,

$$\phi(x_n, x_1) \quad = \quad \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1)$$

Therefore $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (1.2), $\{x_n\}$ is also bounded. Since the space E is reflexive, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ (some point in $C = C_1$). Since C_n is closed and convex and $C_{n+1} \subset C_n$, we see that C_n is weakly closed and $\bar{x} \in C_n, \forall n \geq 1$. Since $x_{n_i} = \Pi_{C_{n_i}} x_1$, we have

$$\phi(x_{n_i}, x_1) \leq \phi(\bar{x}, x_1), \forall n_i \geq 1.$$

On the other hand, from the weakly lower semicontinuity of the norm, we have that

$$\begin{aligned}
\phi(\bar{x}, x_1) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_1\rangle + \|x_1\|^2 \\
&\leq \liminf_{n_i\to\infty}(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1\rangle + \|x_1\|^2 \\
&= \liminf_{n_i\to\infty} \phi(x_{n_i}, x_1) \\
&\leq \limsup_{n\to\infty} \phi(x_{n_i}, x_1) \\
&\leq \phi(\bar{x}, x_1),
\end{aligned}$$

which implies that $\phi(x_{n_i}, x_1) \to \phi(\bar{x}, x_1)$ as $n_i \to \infty$. So $\|x_{n_i}\| \to \|\bar{x}\|$ as $n_i \to \infty$. In view of the Kadec-Klee property of E and $x_{n_i} \rightharpoonup \bar{x}$, we obtain that $x_{n_i} \to \bar{x}$ as $n_i \to \infty$. If there exists another subsequence $\{x_{n_j}\}$ such that $\{x_{n_j}\} \rightharpoonup \bar{y}$, then from Lemma (1.1)(i) we have

$$\begin{aligned}
\phi(\bar{x}, \bar{y}) &= \lim_{n_i\to\infty, n_j\to\infty} \phi(x_{n_i}, x_{n_j}) \\
&= \lim_{n_i\to\infty, n_j\to\infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_1) \\
&\leq \lim_{n_i\to\infty, n_j\to\infty} (\phi(x_{n_i}, x_1) - \phi(\Pi_{C_{n_j}} x_1, x_1)) \\
&= \lim_{n_i\to\infty, n_j\to\infty} (\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1)) \\
&= 0
\end{aligned}$$

That is to say $\bar{x} = \bar{y}$. So

$$\lim_{n\to\infty} x_n = \bar{x}$$

holds. By the way, we get that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \frac{\mu_n M^* \sup_{p\in F(T)} \phi(p, x_n) + \nu_n}{1-\kappa} = 0.$$

(IV) Now we prove $\bar{x} \in F(T)$.

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. By the boundedness of $\{\phi(x_n, x_1)\}$, we know the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of C_n, for any positive integer $m \geq n$, we have $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$. This shows that

$$\phi(x_m, x_n) \quad = \quad \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \to 0, \ as\ m, n \to \infty.$$

When $m = n+1$, we also have $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. It follows from Lemma 1.2 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

It follows from (2.1) and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$ that

$$\phi(x_n, T(PT)^{n-1}x_n) \leq \frac{2}{1-\kappa}\langle x_n - x_{n+1}, Jx_n - JT(PT)^{n-1}x_n\rangle + \xi_n.$$

So we have

$$\lim_{n\to\infty} \phi(x_n, T(PT)^{n-1}x_n) = 0. \tag{2.5}$$

By Lemma 1.1 (ii) and (2.5), we have

$$\lim_{n\to\infty} (\|x_n\| - \|T(PT)^{n-1}x_n\|) = 0. \tag{2.6}$$

Since $x_n \to \bar{x}$, we have

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n\| = \|\bar{x}\|.$$

Since J is uniformly continuous on each bounded subset of E, we have that

$$\lim_{n\to\infty} \|JT(PT)^{n-1}x_n\| = \|J\bar{x}\|. \tag{2.7}$$

This implies that $\{JT(PT)^{n-1}x_n\}$ is bounded. Both E and E^* are reflexive, without loss of generality, we may assume there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $JT(PT)^{n_i-1}x_{n_i} \rightharpoonup y^* \in E^*$. In view of the reflexivity of E, we see that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned}\phi(x_{n_i}, T(PT)^{n_i-1}x_{n_i}) &= \|x_{n_i}\|^2 - 2\langle x_{n_i}, JT(PT)^{n_i-1}x_{n_i}\rangle + \|T(PT)^{n_i-1}x_{n_i}\|^2\\ &= \|x_{n_i}\|^2 - 2\langle x_{n_i}, JT(PT)^{n_i-1}x_{n_i}\rangle + \|J(T(PT)^{n_i-1}x_{n_i})\|^2\end{aligned}$$

By (2.5), taking $\lim_{n_i\to\infty}$ on both sides of the equality above yields that

$$\begin{aligned}0 &= \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2\\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|Jy\|^2\\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|y\|^2\\ &= \phi(\bar{x}, y)\end{aligned}$$

That is $\bar{x} = y$, which in turn implies that $y^* = J\bar{x}$. It follows that $JT(PT)^{n_i-1}x_{n_i} \rightharpoonup J\bar{x}$. By the Kadec-Klee property of E and (2.7), we obtain $\lim_{n_i\to\infty} JT(PT)^{n_i-1}x_{n_i} = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demicontinuous, we have $T(PT)^{n_i-1}x_{n_i} \rightharpoonup \bar{x}$. In view of the Kadec-Klee property of E and (2.6), we have

$$lim_{n_i\to\infty} T(PT)^{n_i-1}x_{n_i} = \bar{x}.$$

If there exists another subsequence $\{x_{n_j}\}$ such that $JT(PT)^{n_j-1}x_{n_j} \rightharpoonup \hat{y}^* \in E^*$, and there exists an element $\hat{y} \in E$ such that $J\hat{y} = \hat{y}^*$, we can similarly prove that $\bar{x} = \hat{y}$ and

$$lim_{n_j\to\infty} T(PT)^{n_j-1}x_{n_j} = \bar{x}.$$

So we see that

$$lim_{n\to\infty} T(PT)^{n-1}x_n = \bar{x}. \tag{2.8}$$

Again by the assumptions that T is asymptotic regular and (2.8), we have

$$\begin{aligned}&\lim_{n\to\infty}\|T(PT)^n x_n-\bar{x}\|\\&\le\lim_{n\to\infty}(\|T(PT)^n x_n-T(PT)^{n-1}x_n\|+\|T(PT)^{n-1}x_n-\bar{x}\|)\\&=0.\end{aligned}\tag{2.9}$$

It follows from (2.9) that

$$\lim_{n\to\infty}T(PT)^n x_n=\bar{x}\ \textit{and}\ \lim_{n\to\infty}TP(T(PT)^{n-1}x_n)=\bar{x}.$$

By virtue of the continuity of TP, we have $TP\bar{x}=\bar{x}$. Since $\bar{x}\in C$ and $P\bar{x}=\bar{x}$, so we get $T\bar{x}=\bar{x}$. So we have $\bar{x}\in F(T)$.

(V) Finally, we prove that $x_n\to\bar{x}=\Pi_{F(T)}x_1$.

Let $\omega=\Pi_{F(T)}x_1$. Since $\omega\in F(T)\subset C_n$ and $x_n=\Pi_{C_n}x_1$, we get $\phi(x_n,x_1)\le\phi(\omega,x_1),n\ge 1$. This implies that

$$\phi(\bar{x},x_1)=\lim_{n\to\infty}\phi(x_n,x_1)\le\phi(\omega,x_1).\tag{2.10}$$

In view of the definition of $\Pi_{F(T)}x_1$, from (2.10) we have $\bar{x}=\omega$. Therefore, $x_n\to\bar{x}=\Pi_{F(T)}x_1$. This completes the proof of *Theorem* 2.1.

Corollary 2.2. *Let E be a real uniformly convex, smooth Banach space, C be a nonempty closed convex subset of E. Let $T:C\to C$ be a continuous asymptotically strict quasi-ϕ pseudocontraction (definition see, Definition 1.1). Suppose T is asymptotically regular and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}x_1\in E, \textit{chosen arbitrarily; } C_1=C\\ C_{n+1}=\{z\in C_n:\phi(x_n,T^n x_n)\le\dfrac{2}{1-\kappa}\langle x_n-z,Jx_n-JT^n x_n\rangle+\xi_n\}\\ x_{n+1}=\Pi_{C_{n+1}}x_1,\forall n\ge 1.\end{cases}\tag{2.11}$$

where $\xi_n=\frac{\mu_n\sup_{p\in F(T)}\phi(p,x_n)}{1-\kappa}$. Then the iterative sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$ which is a fixed point of T in C.

Remark 2.3. Corollary 2.3 is a generalization of the main result in [13].

Corollary 2.4. *Let E be a real uniformly convex and smooth Banach space, C be a nonempty closed convex subset of E. Let $T:C\to E$ be a continuous quasi-ϕ asymptotically nonexpansive nonself mapping. Suppose T is asymptotically regular and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}x_1\in E, \textit{chosen arbitrarily, } C_1=C,\\ C_{n+1}=\{z\in C_n:\phi(x_n,T(PT)^{n-1}x_n)\le 2\langle x_n-z,Jx_n-JT(PT)^{n-1}x_n\rangle+\xi_n\},\\ x_{n+1}=\Pi_{C_{n+1}}x_1,\forall n\ge 1.\end{cases}\tag{2.12}$$

where $\xi_n=(k_n-1)\sup_{p\in F(T)}\phi(p,x_n)$. Then the iterative sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Remark 2.5. It is obvious that if $T:C\to C$ is a continuous quasi-ϕ asymptotically nonexpansive mapping, then the conclusion of Corollary 2.4 still holds.

3. Application to a system of equilibrium problems

Let H be a real Hilbert space, C be a nonempty closed and convex subset of H and $f : C \times C \to \mathscr{R}$ be a bifunction satisfying the following conditions [24]:

(A1) $f(x, x) = 0, \forall x \in C$;

(A2) $f(x, y) + f(y, x) \leq 0, \forall x, y \in C$;

(A3) for each $x, y, z \in C$, $lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(A4) for each given $x \in C$, the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The "$so - called$" equilibrium problem with respect to f is to find a $x^* \in C$ such that $f(x^*, y) \geq 0, \forall y \in C$. The set of its solutions is denoted by $EP(f)$. Let $r > 0$, $x \in H$ and define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \forall x \in H.$$

then

(1) T_r is single-valued, and so $z = T_r x$;

(2) $F(T_r) = EP(f)$, and $F(T_r)$ is a nonempty and closed convex subset of C;

(3) $T_r : C \to C$ is a nonexpansive mapping. Since $F(T_r)$ is nonempty, and so it is a quasi-nonexpansive mapping from C to C, where $\phi(x, y) = \|x - y\|^2, \forall x, y \in H$.

In the following, we shall utilize corollary 2.4 to study an iterative algorithm for a system of equilibrium problems. We have the following result.

Theorem 3.1. *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H. Let $f : C \times C \to \mathscr{R}$ be a bifunction satisfying conditions (A1)–(A4) as given above. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E, \textit{chosen arbitrarily; } C_1 = C \\ f(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, r > 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|x_n - u_n\|^2 \leq 2\langle x_n - z, x_n - u_n\rangle + \xi_n\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \forall n \geq 1. \end{cases} \tag{3.1}$$

where $P_{C_{n+1}}$ is the metric projection from H onto C_{n+1}. If $\mathscr{F} = F(T_r) \neq \emptyset$, then $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_1$.

Proof. Since $u_n = T_r x_n$ and $F(T_r) = EP(f)$ is nonempty closed and convex. Again since T_r is a nonexpansive mapping, $F(T_r)$ is nonempty, and so T_r is quasi-ϕ nonexpansive mapping. Hence (3.1) can be rewritten as follows

$$\begin{cases} x_1 \in H, \text{chosen arbitrarily; } C_1 = C \\ C_{n+1} = \{z \in C_n : \|x_n - T_r x_n\|^2 \leq 2\langle x_n - z, x_n - T_r x_n\rangle + \xi_n\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \forall n \geq 1. \end{cases} \tag{3.2}$$

The conclusion of Theorem 3.1 can be obtained from Corollary 2.4 immediately.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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