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RICH DYNAMICS OF AN SIRS EPIDEMIC MODEL WITH DISEASE RELATED DEATH RATE

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Abstract In this paper, an SIRS epidemic model with an asymptotically homogeneous transmission function and disease related death rate is formulated and studied. We obtain the disease free and endemic equilibrium. We prove the global stability for equilibriums. We give an example to illustrate the main results.

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1. Introduction

Mathematical epidemiology has become a interesting subject of research work since the seminal model of Kermack and Mckendric [24] on SIRS (susceptible-infected-recovered-susceptible) system, in which a disease is transmitted upon contact has been thoroughly investigated. Results for the simplest epidemiological models are given in Hethcote [8].The different epidemic models have been proposed and studied by many authors (see, for example, Lie et al. [9, 10], Capasso and Serio [2], Hethcote et al. [7], Derrick and Vanden Driessche [4], Beretta et al. [1], Song et al. [15], Ruan and Wang [14], D'onofno et al. [5], Xiao and Ruan [16]. Pathak et al. [13] have considered an SIR epidemic model with an asymptotically homogeneous transmission function. Mehta et al. [10] proposed the epidemic model with an asymptotically homogeneous transmission function. Recently Nirwani et al. [12] describe the stability analysis of an SIR model with immunity and modified transmission function. The rate at which susceptible becomes infectious is called

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the incidence in an epidemiological model. If the unit time is days, then the incidence is the number of new infection per day. The average number of secondary infections produced by one infected individuals during the mean course of infection (infectious period) in a completely susceptible population is called a basic reproductive number or simply the reproductive number σ. If $\sigma < 1$, then on average, the number of new infection by one infected individual over the mean course of the disease (infectious period) is less than one which implies that the disease dies out. If $\sigma > 1$, then the number of new infections produced by one infected individual is greater than one which implies that disease persist.

In this paper we consider an SIRS epidemic model with an asymptotically homogeneous transmission function and disease related death rate. In the next section, we present the model and derive the disease free equilibrium and the endemic equilibrium. In the third section, we prove some theorems for the global stability of the disease free and endemic equilibrium. The fourth section contains an example which demonstrates validity of main result. In the last section, we give conclusion.

2. Formulation of an SIRS model

We formulate an SIRS model with an asymptotically homogeneous transmission function $\frac{\beta XY}{1+aX+bY}$ and disease related death rate α. Let $X(t)$ is the number of susceptible individuals, $Y(t)$ is the number of infective individuals, $Z(t)$ is the number of recovered individuals, and $N(t)$ is the total population size, the transfer diagram is

$$\xrightarrow{A} X \xrightarrow{\frac{\beta XY}{1+aX+bY}} Y \xrightarrow{\gamma Y} Z \xrightarrow{\delta Z} X$$
$$\downarrow dX \qquad \downarrow (\alpha+d)Y \qquad \downarrow dZ$$

The parameters in the model are:

A=constant immigration rate
d=natural death rate constant
β=transmission coefficient
α=disease-related death rate constant
γ=recovery rate constant
δ=loss of immunity rate constant, a and b are the parameters which measure the effects of sociological, psychological or other mechanisms. We assume that d, α and δ are non-negative and that A, β, γ and $\delta + d$ are positive.

The nonlinear ordinary differential equations corresponding to the transfer diagram are:

$$\begin{aligned}
\frac{dX}{dt} &= A - \frac{\beta XY}{1+aX+bY} - dX + \delta Z \\
\frac{dY}{dt} &= \frac{\beta XY}{1+aX+bY} - (\gamma + \alpha + d)Y \\
\frac{dZ}{dt} &= \gamma Y - (\delta + d)Z \\
\frac{dN}{dt} &= A - dN - \alpha Y
\end{aligned} \tag{2.1}$$

Where $N = X+Y+Z$. In the absence of disease i.e. $\alpha = 0$ the population size approaches the constant size $\frac{A}{d}$. For the asymptotically transmission function the contact number or

basic reproduction number is

$$\sigma = \frac{\beta A - Aa(\gamma + \alpha + d)}{d(\gamma + \alpha + d)} \tag{2.2}$$

For the system (2.1) the first octant in XYZ space is positively invariant. Because $\frac{dN}{dt} < 0$ for $N > \frac{A}{d}$, all paths in the first octant approach, enter or stay inside the subset $T = \left\{ \left(X, Y, Z\right) : X + Y + Z \leq \frac{A}{d} \right\}$. The continuity of the right side of (2.1) and its derivatives implies that unique solutions exits on a maximal time interval. Since solutions approach, enter or stay in T, they are eventually bounded and hence exist for all positive time [3]. We first consider the existence of equilibrium of system (2.1).

For any values of parameter, model (2.1) always has a disease-free equilibrium $P_0 = \left(\frac{A}{d}, 0, 0\right)$. To find the positive equilibria, set

$$\begin{aligned} A - \frac{\beta XY}{1 + aX + bY} - dX + \delta Z &= 0 \\ \frac{\beta XY}{1 + aX + bY} - (\gamma + \alpha + d)Y &= 0 \\ \gamma Y - (\delta + d)Z &= 0 \\ A - dN - \alpha Y &= 0 \end{aligned} \tag{2.3}$$

3. Main Results

Theorem 3.1: From the system (2.2) it follows that

(i) if $\sigma \leq 1$, then there is no positive equilibrium;
(ii) if $\sigma > 1$, then there is a unique positive equilibrium $P_e = \left(X_e, Y_e, Z_e\right)$ of the system (2.1), called the "endemic equilibrium", given by

$$\begin{aligned} X_e &= \frac{(\gamma + \alpha + d)(1 + bY_e)}{\beta - a(\gamma + \alpha + d)} \\ Y_e &= \frac{(\delta + d)\left[A\beta - (\gamma + \alpha + d)(Aa + d)\right]}{bd(\gamma + \alpha + d)(\delta + d) + [\alpha(\delta + d) + d(\gamma + \delta + d)][\beta - a(\gamma + \alpha + d)]} \\ Z_e &= \frac{\gamma Y_e}{\delta + d} \\ N_e &= \frac{A - \alpha Y_e}{d} \end{aligned} \tag{3.1}$$

It is clear that the limit set of system (2.1) is on the plane $X + Y + Z = \frac{A}{d}$. Thus we focus on the reduced system

$$\begin{aligned} \frac{dY}{dt} &= \frac{d\beta Y}{(d + aA) + (b - a)dY - adZ}\left(\frac{A}{d} - Y - Z\right) - (\gamma + \alpha + d)Y \equiv P(Y, Z) \\ \frac{dZ}{dt} &= \gamma Y - (\delta + d)Z \equiv Q(Y, Z) \end{aligned} \tag{3.2}$$

Theorem 3.2: System (3.2) does not have nontrivial periodic orbits if $(2d+\gamma+\delta+\alpha)(b-a) > a\gamma$.

Proof. Since $Y > 0$ and $Z > 0$. Take a Dulac function

$$D(Y,Z) = \frac{(d+aA)+(b-a)dY - adZ}{d\beta Y}.$$

We have

$$\frac{\partial(DP)}{\partial Y} + \frac{\partial(DQ)}{\partial Z} = -1 - \frac{(\delta+d)(d+aA)}{d\beta Y} - \left[(2d+\gamma+\delta+\alpha)(b-a) - a\gamma\right] < 0$$

if $(2d+\gamma+\delta+\alpha)(b-a) > a\gamma$

In order to study the properties of the disease-free equilibrium P_0 and the endemic equilibrium P_e.

Theorem 3.3: The equilibrium $P_0 = \left(\frac{A}{d}, 0, 0\right)$ is locally asymptotically stable if $\sigma \leq 1$ and P_0 is saddle point if $\sigma > 1$.

Proof. The Jacobian of system (2.1) at P_0 is

$$J(P_0) = \begin{pmatrix} -d & -\frac{\beta A}{d+aA} & \delta \\ 0 & \frac{\beta A}{d+aA} - (\gamma+\alpha+d) & 0 \\ 0 & \gamma & -(\delta+d) \end{pmatrix}$$

The characteristic equation is

$$(d+t)(\delta+d+t)\left[\frac{\beta A}{d+aA} - (\gamma+\alpha+d) - t\right] = 0 \tag{3.3}$$

The roots of (3.3) are

$$-d,\ -(\delta+d) \text{ and } \frac{\beta A}{d+aA} - (\gamma+\alpha+d)$$

The first two roots having negative real parts and third root $\frac{\beta A}{d+aA} - (\gamma+\alpha+d)$ will have negative real part if $\sigma \leq 1$. Thus all roots of (3.3) have negative real parts so P_0 is locally asymptotically stable if $\sigma \leq 1$ and the root $\frac{\beta A}{d+aA} - (\gamma+\alpha+d)$ will have positive real part if $\sigma > 1$ so P_0 is saddle point.

Theorem 3.4: The equilibrium $P_0 = \left(\frac{A}{d}, 0, 0\right)$ is globally asymptotically stable if $\sigma \leq 1$.

Proof. Since the set $T = \left\{(X,Y,Z) : X+Y+Z \leq \frac{A}{d}\right\}$ is attractive and positive invariant.

To prove that all paths in T approach $P_0 = \left(\frac{A}{d}, 0, 0\right)$ for $\sigma \leq 1$, define the Liapunov function $L = Y$ in T with

$$\frac{dL}{dt} = \frac{dY}{dt} = \left[\frac{\beta XY}{1 + aX + bY} - (\gamma + \alpha + d)\right] Y \leq 0. \tag{3.4}$$

The Lasalle-Liapunov theory [6] implies that all paths in T approach the largest positively invariant subset of the set T where $\frac{dL}{dt} = 0$.

Here $\frac{dL}{dt} = 0$ only if $Y = 0$ or $(X, Y, Z) = P_0$. The positively invariant subset of the plane $Y = 0$ is the point P_0 so P_0 is globally asymptotically stable for $\sigma \leq 1$.

To study the properties of the endemic equilibrium P_e. Let us define

$$x = \frac{\beta}{\delta + d} Y,\ y = \frac{\beta}{\delta + d} Z,\ \tau = (\delta + d)t$$

We obtain

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{px}{1 + qx - ry}(K - x - y) - mx, \\ \frac{dy}{d\tau} &= sx - y, \end{aligned} \tag{3.5}$$

Where

$$\begin{aligned} p &= \frac{d}{d + aA},\ q = \frac{(\delta + d)d(b - a)}{\beta(d + aA)},\ r = \frac{a(\delta + d)d}{\beta(d + aA)}, \\ k &= \frac{A\beta}{d(\delta + d)},\ m = \frac{\gamma + \alpha + d}{\delta + d},\ s = \frac{\gamma}{\delta + d}. \end{aligned}$$

For equilibrium point set,

$$\frac{dx}{d\tau} = 0 \text{ and } \frac{dy}{d\tau} = 0$$

We obtain, two equilibrium point $(0, 0)$ and (x_e, y_e) where

$$x_e = \frac{Kp - m}{p(1 + s) + m(q - rs)},\ y_e = sx_e$$

The trivial solution $(0, 0)$ of system (3.5) is the disease-free equilibrium P_0 of model (2.1) and the unique positive equilibrium (x_e, y_e) of system (3.5) is the endemic equilibrium P_e of model (2.1) if and only if $Kp - m > 0$ and $q - rs > 0$.

Theorem 3.5: Suppose $m - Kp < 0$, then there is a unique endemic equilibrium (x_e, y_e) of model (3.5) which is a stable node.

Proof. The Jacobian of system (3.5) at (x_e, y_e) is

$$J = \begin{pmatrix} \frac{px_e[sx_e(r+q) - (1+Kq)]}{(1+qx_e - rsx_e)^2} & \frac{px_e[(Kq-1) - x_e(r+q)]}{(1+qx_e - rsx_e)^2} \\ s & -1 \end{pmatrix}$$

$$\det J = \frac{px_e[(1 + s) + K(q - rs)]}{(1 + qx_e - rsx_e)^2}$$

Since $q > rs, \det(J) > 0$ when $m - Kp < 0$ and

$$tr(J) = \frac{[ps(r+q)x_e - p(1+Kq)]x_e - [x_e(rs-q)-1]^2}{(1+qx_e - rsx_e)^2}$$

The sign of $tr(J)$ is determined by

$$S = [ps(r+q)x_e - p(1+Kq)]x_e.$$

Substituting $x_e = \frac{Kp-m}{p(1+s)+m(q-rs)}$ into S, We have

$$S = \frac{p[-K(p+mq)(q-rs) - (mqs+mq+p+ps)](Kp-m)}{[p(1+s)+m(q-rs)]^2}.$$

Since $q > rs, [p(1+s)+m(q-rs)]^2 > 0$ and $[-K(p+mq)(q-rs)-(mqs+mq+p+ps)] < 0$, hence $S < 0$ if $m - Kp < 0$. However, when $m - Kp < 0$, we have $tr(J) < 0$. This completes the proof.

Theorem 3.6: The equilibrium $P_e = \left(X_e, Y_e, Z_e\right)$ is globally asymptotically stable if $\sigma > 1$.

Proof. The proof can be obtained by theorem 4.5.

4. Numerical

In this section, we give an example to demonstrate the results obtained in the previous sections.

Example 4.1. We take the parameters of the system as $d = 2.27, a = 3.4, b = 4, A = 3.4, \delta = 1, \alpha = 0.17, \beta = 9.2, \gamma = 0.18$. Then $P_0 = (1.4977, 0, 0)$ and $\sigma = 0.1666 < 1$. Therefore, by theorem 3.4, P_0 is a global asymptotically stable in the first octant.
Now we take the parameter of the system as $d = 0.27, a = 3.4, b = 4, A = 3.4, \delta = 1, \alpha = 0.17, \beta = 4, \gamma = 0.18$. Then $P_e = (6.9476, 5.0504, 0.7158)$ and $\sigma = 38.42 > 1$. Therefore, by theorem 3.6, P_e is a global asymptotically stable in the interior of the first octant.

5. Conclusion

In this paper, we have investigated the rich dynamics SIRS epidemic model with an asymptotically homogenous transmission function and disease related death rate. This model is assumed that the dynamics of the population is governed by the exponential birth and death. We have proved that, if $\sigma > 1$ then the disease-free equilibrium P_0 is unstable and there exists uniquely an endemic equilibrium. On the other hand, if $\sigma \leq 1$ then P_0 is globally asymptotically stable. Furthermore, we have proved that if $\sigma > 1$ then the endemic equilibrium $P_e = \left(X_e, Y_e, Z_e\right)$ is globally asymptotically stable.

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