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COMMON FIXED POINT OF THREE-STEP ITERATION WITH ERRORS FOR NONSELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract In this paper, we introduce and study the modified a new iteration process with errors approximating the common fixed point for three asymptotically nonexpansive mappings nonself in a real uniformly convex and smooth Banach space with P as a sunny nonexpansive retraction. The results obtained in this paper extend and improve the recent ones announced by Xu and Noor [22] and many others.

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1. Introduction

Let K be a nonempty subset of a real normed linear space E. A mapping $T : K \to K$ is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in K$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|, \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$. A mapping $T : K \to K$ is called uniformly L-Lipschitzian if there exists constant $L > 0$ such that

$$\| T^n x - T^n y \| \leq L \| x - y \|, \tag{1.2}$$

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for all $x, y \in K$ and $n \geq 1$. Also T is called asymptotically quasi-nonexpansive if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that for all $x \in K$, the following inequality holds:

$$\|T^n x - x^*\| \leq (1 + k_n)\|x - x^*\|, \ \forall x^* \in F(T), \ n \geq 1. \tag{1.3}$$

From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive. Every asymptotically nonexpansive mapping with a fixed point is asymptotically quasi-nonexpansive, but the converse may be not true.

In 1972, Goebel and Kirk [10] introduce the class of asymptotically nonexpansive self-mapping, they prove that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping on K, then T has a fixed point.

In 2003, Chidume et al.[4] introduce the concept of nonself-asymptotically nonexpansive mapping as the generalization of asymptotically nonexpansive self-mapping.

2. Preliminaries

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to K$ is said to be a retraction if $P^2 = P$ It follows that if a map P is a retract, then $Py = y$ for all y in the range of P. As the generalization of an asymptotically nonexpansive self-mapping, Chidume et al.[4] introduced a nonself-asymptotically nonexpansive mapping as follows:

Definition 2.1. [4]. Let K be a nonempty subset of real normed linear space E. Let $P : E \in K$ be the nonexpansive retraction of E into K. A nonself-mapping $T : K \to E$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, 1)$ with $k_n \to 0$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq (1 + k_n)\|x - y\|, \ \forall x, y \in K, \ n \geq 1. \tag{2.1}$$

A nonself mapping $T : K \to E$ is said to be uniformly $L-$ Lipschitzian if there exists a constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \ \forall x, y \in K, \ n \geq 1. \tag{2.2}$$

If T is self-mapping, then P becomes the identity mapping, so that (2.2) reduces to (1.1). In [4], they studied the following iterative algorithm:

$$x_1 \in K, \, x_{x+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \ \forall \ n \geq 1. \tag{2.3}$$

Chidume et al.[4] established demiclosed principle, strong and weak convergence theorems for nonself-asymptotically nonexpansive mapping in uniformly convex Banach space. Recently concerning the convergence problem of an explicit iterative process to a common fixed point for some sonself-asymptotically nonexpansive mapping in uniformly convex Banach space have been considered by several authors (see, for example, [2-8] and the reference therein). Recently, Zhou et al.[1] introduced the following definition.

Definition 2.2. [1]. Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself-mapping $T : K \to E$ is said to be asymptotically nonexpansive with respect to P if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 0$ as $n \to \infty$ such that

$$\| T(PT)^n x - T(PT)^n y \| \leq (1 + k_n) \| x - y \|, \ \forall x, y \in K, \ n \geq 1. \tag{2.4}$$

T is said to be uniformly $L-$ Lipschitzian with respect to P if there exists a constant $L > 0$ such that

$$\| T(PT)^n x - T(PT)^n y \| \leq (1 + k_n) \| x - y \|, \ \forall x, y \in K, \ n \geq 1. \tag{2.5}$$

Remark 2.3. [1]. If $T : K \to E$ is an asymptotically nonexpansive in the light of (2.4) and $P : E \to K$ is a nonexpansive retraction, then for all $x, y \in K, n \geq 1$. We have

$$\begin{aligned} \| T(PT)^n x - T(PT)^n y \| &= \| PT(PT)^{n-1} x - PT(PT)^{n-1} y \| \\ &\leq \| T(PT)^{n-1} x - T(PT)^{n-1} y \| \\ &\leq k_n \| x - y \| . \end{aligned} \tag{2.6}$$

But, the converse may not be true. Actually they studied the iteration algorithm

$$x_1 \in K, x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, n \geq 1, \tag{2.7}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[b, 1-b]$ for some $b \in (0, 1)$ which satisfy conditions $\alpha_n + \beta_n + \gamma_n = 1$ and T_1, T_2 are nonself-asymptotically nonexpansive mapping with respect to P. Zhou et al.[1] studied the strong and weak convergence theorem for nonself-asymptotically nonexpansive mapping with respect to P.

In 2015, Gunduz and Akbulut [9] introduced the following three-step iteration process. Let E be a normed space, K be a nonempty convex subset of E, Let $T_i : K \to E (i = 1, 2, 3)$ be nonself-asymptotically nonexpansive mapping with respect to P. Then for a given $x_1 \in K$ and $n \geq 1$, compute the iterative sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 + a_{n1} - b_{n1}) x_n + a_{n1} (PT_1)^n y_{n+1} + b_{n1} (PT_2)^n y_n, \\ y_{n+1} = (1 + a_{n2} - b_{n2}) x_n + a_{n2} (PT_2)^n y_n + b_{n2} (PT_3)^n x_n, \\ y_n = (1 + a_{n3}) x_n + a_{n3} (PT_3)^n x_n, \end{cases} \tag{2.8}$$

for $n \geq 1$ where $\{a_{ni}\}, \{b_{ni}\}$ and $\{1 - a_{ni} - b_{ni}\}$ are sequence in $[0, 1]$ for all $i = 1, 2, 3$.

The main purpose of this paper is to construct an iteration process (2.9) below for common fixed points of three nonself-asymptotically nonexpansive mapping and to prove some strong and weak convergence theorem for such mappings in uniformly convex Banach spaces.

Let E be a real normed linear space, and K be a nonempty closed convex subset of E which is also a nonexpansive retraction of E with a retraction P. Let $T_i : K \to E (i = 1, 2, 3)$ be nonself-asymptotically nonexpansive mappings with respect to P. Then for a given $x_1 \in K$ compute the iterative sequences $\{x_n\}$ defined by,

$$\begin{cases} x_{n+1} = (1 - \alpha_{n1} - \beta_{n1} - \gamma_{n1}) x_n + \alpha_{n1} (PT_1)^n y_{n+1} + \beta_{n1} (PT_2)^n y_n + \gamma_{n1} \mu_{n1}, \\ y_{n+1} = (1 - \alpha_{n2} - \beta_{n2} - \gamma_{n2}) x_n + \alpha_{n2} (PT_2)^n y_n + \beta_{n2} (PT_3)^n x_n + \gamma_{n2} \mu_{n2}, \\ y_n = (1 - \alpha_{n3} - \gamma_{n3}) x_n + \alpha_{n3} (PT_3)^n x_n + \gamma_{n3} \mu_{n3}, \end{cases} \tag{2.9}$$

for $n \geq 1$, where $\{\alpha_{nk}\}, \{\beta_{ni}\}, \{\gamma_{ni}\}$ and $\{1 - \alpha_{ni} - \beta_{ni} - \gamma_{ni}\}$ are sequence in $[0, 1]$ and $\sum_{n=1}^{\infty} \gamma_{ni} < \infty$ satisfying certain conditions μ_{ni} are bounded sequcences in K, for $i = 1, 2, 3$.

If $\gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the iteration defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_{n1} - \beta_{n1}) x_n + \alpha_{n1} (PT_1)^n y_{n+1} + \beta_{n1} (PT_2)^n y_n, \\ y_{n+1} = (1 - \alpha_{n2} - \beta_{n2}) x_n + \alpha_{n2} (PT_2)^n y_n + \beta_{n2} (PT_3)^n x_n, \\ y_n = (1 - \alpha_{n3}) x_n + \alpha_{n3} (PT_3)^n x_n. \end{cases} \tag{2.10}$$

If $\beta_{n1} = \beta_{n2} = 0$ and $\gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the iteration defined by

$$\begin{cases} x_{n+1} = (1-\alpha_{n1})x_n + \alpha_{n1}(PT_1)^n y_{n+1} + \beta_{n1}(PT_2)^n z_n, \\ y_{n+1} = (1-\alpha_{n2})x_n + \alpha_{n2}(PT_2)^n y_n + \beta_{n2}(PT_3)^n x_n, \\ y_n = (1-\alpha_{n3})x_n + \alpha_{n3}(PT_3)^n x_n, \qquad \text{for } n \geq 1. \end{cases} \tag{2.11}$$

If $\alpha_{n3} = \alpha_{n2} = \beta_{n2} = 0$ and $\gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the iteration defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(PT_1)^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n(PT_2)^n x_n, \\ x_1 \in K, \qquad \text{for } n \geq 1. \end{cases} \tag{2.12}$$

If $\alpha_{n3} = \alpha_{n1} = \beta_{n2} = 0$ and $\gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the iteration (2.7) defined by Zhou et al.[1].

If $\alpha_{n3} = \alpha_{n2} = \beta_{n2} = \beta_{n1} = \gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ and $T_1 = T_2 = T$ for all $n \geq 1$, then (2.9) reduces to the iteration (2.3).

If $T_1 = T_2 = T_3 = T$ are self-mapping and $\beta_{n2} = \beta_{n1} = \gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the Noor iteration defined by Xu and Noor [22]

$$\begin{cases} x_{n+1} = (1-\alpha_{n1})x_n + \alpha_{n1}T^n y_{n+1}, \\ y_{n+1} = (1-\alpha_{n2})x_n + \alpha_{n2}T^n y_n, \\ y_n = (1-\alpha_{n3})x_n + \alpha_{n3}T^n x_n, \qquad \text{for } n \geq 1, \end{cases} \tag{2.13}$$

where $\{\alpha_{ni}\}$ are sequence in $[0,1]$ for all $i = 1,2,3$.

If $T_1 = T_2 = T$ are self-mapping and $\alpha_{n3} = \beta_{n1} = \beta_{n2} = \gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the modified Ishikawa iteration process [14]

$$\begin{cases} x_{n+1} = (1-\alpha_{n1})x_n + \alpha_{n1}T^n y_{n+1}, \\ y_{n+1} = (1-\alpha_{n2})x_n + \alpha_{n2}T^n y_n, \qquad \text{for } n \geq 1. \end{cases} \tag{2.14}$$

If $T_1 = T$ are self-mapping and $\alpha_{n3} = \beta_{n1} = \beta_{n2} = \gamma_{n1} = \gamma_{n2} = \gamma_{n3} = 0$ for all $n \geq 1$, then (2.9) reduces to the modified Mann iteration process [21]

$$\{ \; x_{n+1} = (1-\alpha_{n2})x_n + \alpha_{n2}T^n y_n, \quad \text{for } n \geq 1, \tag{2.15}$$

where $\{\alpha_{n1}\}$ are sequences in $[0,1]$.

Now we list the following definitions and results which are useful in the sequel.

Let E be Banach space with $dim(E) \geq 2$, the modulus of E is the function

$$\delta_E(\varepsilon) = \inf\left\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\right\} \tag{2.16}$$

The Banach space E is uniformly convex if and only if with $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,]$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. The space E said to be smooth if

$$\lim_{t\to\infty} \frac{\|x+ty\| - \|x\|}{t} \tag{2.17}$$

exists for all $x, y \in S(E)$.

Let C, D be nonempty subset of a Banach space E such that C in nonempty closed convex and $D \subset C$, A retraction $P : C \to D$ is sad to be sunny [12] if $P(Px + t(x - Px)) = Px$ for all $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also a nonexpansive mapping.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demi-closed at p, if for each sequence $\{x_n\}$ in $D(T)$, the conditions $x_n \to x_0$ weakly and $Tx_n \to p$ strongly imply $Tx_0 = p$.

A mapping $T : K \to K$ is said to be completely continuous if for every bounded sequence $\{x_n\}$, there exists a subsequence say $\{x_{nj}\}$ of $\{x_n\}$ such that $\{Tx_{nj}\}$ converges to some element of the range T.

A mapping $T : K \to K$ is said to demi-compact if any sequence $\{x_n\}$ in K satisfying $x_n - Tx_n \to 0$ as $n \to \infty$ has a convergent subsequence.

Recall that the mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [11] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$. Senter and Dotson [11] pointed out that every continuous and demi-compact mapping must satisfy Condition (A). Different modifications of the Condition (A) for two finite families of self-maps have been made recently in the literature [19], [21]. Yang and Xie [8] modified these condition for three nonself-asymptotically nonexpansive mapping $T_i : K \to E\,(i = 1, 2, 3)$ as follows:

Mappings $T_i : K \to E\,(i = 1, 2, 3)$ wh the nonempty common fixed point set in K is said to satisfy Condition (B) with respect to the sequence $\{u_n\}$ if there is nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that for all $n \geq 1$. We know that Condition (B) is weaker than the demi-compactness of mappings $T_i (i = 1, 2, 3)$.

Lemma 2.4. *[17] Let $\{a_n\}, \{b_n\}, \{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \tag{2.18}$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} a_n < \infty$ then,
(i) $\lim_{n\to\infty} a_n$ exists,
(ii) In particular, if $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[16] Let E be a uniformly convex Banach space and $B_R = \{x \in E : \|x\| < R\}$, $R > 0$. Then there exists a continuous, strictly increasing ad convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|), \tag{2.19}$$

for all $x, y, z, w \in B_R$ and $\alpha, \beta, \gamma, \lambda \in [0, 1]$ with $\alpha + \beta + \gamma + \lambda = 1$.

Lemma 2.6. *[18] Let E be real smooth Banach space, let k be nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

Lemma 2.7. *[1] Let E be a real smooth and uniformly convex Banach space, K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T : K \to E$ be a weakly inward and asymptotically noexpansive mapping with respect to P with the sequence $\{k_n\} \in [1, \infty)$ such that $\{k_n\} \to 1$ as $n \to \infty$. Then $I - T$ is demi-closed at zero.*

3. Main Results

In this section, we prove our main theorem.

Theorem 3.1. *Let E be a real space and K be a nonempty closed convex subset of E which is also a nonexpansive respect of E. Let $T_i : K \to E (i = 1, 2, 3)$ be three nonself-asymptotically nonexpansive mapping with respect to P with sequences $\left\{t_n^{(i)}\right\}$ such that $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ and $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$. For an arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.9). Then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for $x^* \in \mathcal{F}$.*

Proof. Let $x \in \mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$, by the boundedness of sequences μ_{ni} for $i = 1, 2, 3$, there exists a constant $M_{x^*} > 0$ such that $\|\mu_{ni} - x^*\| \leq M_{x^*}$. Setting $t_n = \max\{t_n^{(1)}, t_n^{(2)}, t_n^{(3)}\}$, Since $\sum_{n=1}^{\infty} t_n < \infty$. From (2.9), we get

$$\begin{aligned}
\|y_n - x^*\| &= \|(1 - \alpha_{n3} - \gamma_{n3})x_n + \alpha_{n3}(PT_3)^n x_n + \gamma_{n3}\mu_{n3} - x^*\| \\
&\leq (1 - \alpha_{n3} - \gamma_{n3})\|x_n - x^*\| + \alpha_{n3}\|(PT_3)^n x_n - x^*\| + \gamma_{n3}\|\mu_{n3} - x^*\| \\
&\leq (1 - \alpha_{n3} - \gamma_{n3})\|x_n - x^*\| + \alpha_{n3} t_n\|x_n - x^*\| + \gamma_{n3}\|\mu_{n3} - x^*\| \\
&\leq \|x_n - x^*\| + \alpha_{n3} t_n\|x_n - x^*\| + \gamma_{n3}\|\mu_{n3} - x^*\| \\
&\leq \|x_n - x^*\| + t_n\|x_n - x^*\| + \gamma_{n3}\|\mu_{n3} - x^*\| \\
&\leq (1 + t_n)\|x_n - x^*\| + \gamma_{n3} M_{x^*} \qquad (3.1)
\end{aligned}$$

By (2.9) and (3.1), we obtain

$$\begin{aligned}
\|y_{n+1} - x^*\| &= \|(1 - \alpha_{n2} - \beta_{n2} - \gamma_{n2})x_n + \alpha_{n2}(PT_2)^n y_n \\
&\quad + \beta_{n2}(PT_3)^n x_n + \gamma_{n2}\mu_{n2} - x^*\| \\
&\leq (1 - \alpha_{n2} - \beta_{n2} - \gamma_{n2})\|x_n - x^*\| + \alpha_{n2}\|(PT_2)y_n - x^*\| \\
&\quad + \beta_{n2}\|(PT_3)x_n - x^*\| + \gamma_{n2}\|\mu_{n2} - x^*\| \\
&\leq (1 - \alpha_{n2} - \beta_{n2} - \gamma_{n2})\|x_n - x^*\| + \alpha_{n2} t_n\|y_n - x^*\| \\
&\quad + \beta_{n2} t_n\|x_n - x^*\| + \gamma_{n2}\|\mu_{n2} - x^*\| \\
&\leq \|x_n - x^*\| + \alpha_{n2} t_n\|y_n - x^*\| \\
&\quad + \beta_{n2} t_n\|x_n - x^*\| + \gamma_{n2}\|\mu_{n2} - x^*\| \\
&\leq \|x_n - x^*\| + t_n\|y_n - x^*\| \\
&\quad + t_n\|x_n - x^*\| + \gamma_{n2}\|\mu_{n2} - x^*\| \\
&\leq \|x_n - x^*\| + t_n\big[(1 + t_n)\|x_n - x^*\| + \gamma_{n3} M_{x^*}\big] \\
&\quad + t_n\|x_n - x^*\| + \gamma_{n2} M_{x^*} \\
&= (1 + 2t_n + t_n^2)\|x_n - x^*\| + (\gamma_{n2} + \gamma_{n3} t_n) M_{x^*} \qquad (3.2)
\end{aligned}$$

Using (2.9), (3.1) and (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1 - \alpha_{n1} - \beta_{n1} - \gamma_{n1})x_n + \alpha_{n1}(PT_1)^n y_{n+1} \\
&\quad + \beta_{n1}(PT_2)^n y_n + \gamma_{n1}\mu_{n1} - x^*\| \\
&\leq \|(1 - \alpha_{n1} - \beta_{n1} - \gamma_{n1})\|x_n - x^*\| + \alpha_{n2}\|(PT_1)^n y_{n+1} - x^*\|
\end{aligned}$$

$$\begin{aligned}
&+\beta_{n1}\|(PT_2)y_n - x^*\| + \gamma_{n1}\|\mu_{n1} - x^*\| \\
&\leq \|(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})\|x_n - x^*\| + \alpha_{n1}t_n\|y_{n+1} - x^*\| \\
&\quad + \beta_{n1}t_n\|y_n - x^*\| + \gamma_{n1}\|\mu_{n1} - x^*\| \\
&\leq \|x_n - x^*\| + \alpha_{n1}t_n\|y_{n+1} - x^*\| \\
&\quad + \beta_{n1}t_n\|y_n - x^*\| + \gamma_{n1}\|\mu_{n1} - x^*\| \\
&\leq \|x_n - x^*\| + t_n\|y_{n+1} - x^*\| + t_n\|y_n - x^*\| + \gamma_{n1}\|\mu_{n1} - x^*\| \\
&\leq \|x_n - x^*\| + t_n\big[(1+2t_n+t_n^2)\|x_n - x^*\| + (\gamma_{n2}+\gamma_{n3}t_n)M_{x^*}\big] \\
&\quad + t_n\big[(1+t_n)\|x_n - x^*\| + \gamma_{n3}M_{x^*}\big] + \gamma_{n1}M_{x^*} \\
&= (1+2t_n+3t_n^2+t_n^3)\|x_n - x^*\| \\
&\quad + (\gamma_{n1}+\gamma_{n2}t_n+\gamma_{n3}t_n+\gamma_{n3}t_n^2)M_{x^*} \qquad (3.3)
\end{aligned}$$

Let $\delta_n = 2t_n + 3t_n^2 + t_n^3$ and $\lambda_n = (\gamma_{n1}+\gamma_{n2}t_n+\gamma_{n3}t_n+\gamma_{n3}t_n^2)M_{x^*}$, we get

$$\|x_{n+1} - x^*\| \leq (1+\delta_n)\|x_n - x^*\| + \lambda_n. \qquad (3.4)$$

Since $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\lambda_n < \infty$. It follows from Lemma 2.4 and (3.4) that the limit $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in \mathcal{F}$. ∎

Theorem 3.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E which is also a nonexpansive respect of E. Let $T_i : K \to E (i = 1, 2, 3)$ be three nonself-asymptotically nonexpansive mapping with respect to P with sequences $\left\{t_n^{(i)}\right\}$ such that $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$. Suppose $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and closed. For an arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.9). Then sequece $\{x_n\}$ as defined by (2.9) converges strongly to a common fixed point of T_i for $(i = 1, 2, 3)$ if and only if*

$$\liminf_{n\to\infty} d\,(x_n, \mathcal{F}) = 0, \qquad (3.5)$$

where $d(x_n, \mathcal{F}) = \inf\{\|x_n - x^*\| : x^* \in \mathcal{F}\}$.

Proof. For any given $x^* \in \mathcal{F}$, we setting $M = \sup_{n\geq 1}\|x_n - x^*\|$. Hence we can rewrite (3.4) as follow

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + M\delta_n + \lambda_n,\ n \geq 1. \qquad (3.6)$$

For any positive integer $m, n \geq 1$, we get

$$\begin{aligned}
\|x_{m+n}\| &\leq \|x_{n+m-1} - x^*\| + M\delta_{n+m-1} + \lambda_{n+m-1} \\
&\leq \|x_{n+m-2} - x^*\| + M(\delta_{n+m-1} + \delta_{n+m-2}) + (\lambda_{n+m-1} + \lambda_{n+m-2}) \\
&\ \ \vdots \\
&\leq \|x_n - x^*\| + M\sum_{i=n}^{n+m-1}\delta_i + \sum_{i=n}^{n+m-1}\lambda_i.
\end{aligned} \qquad (3.7)$$

By (3.4), we get

$$d(x_{n+1},\mathcal{F}) \leq (1+\delta_n)d(x_n,\mathcal{F}) + \lambda_n. \tag{3.8}$$

We applying Lemma 2.4 to (3.8), we obtain the existence of the limit $d(x_n,\mathcal{F})$. By condition (3.5), we get

$$\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0. \tag{3.9}$$

Next, we show that the sequence $\{x_n\}$ converges to a common fixed point of T_i(i=1,2,3). First, we show that $\{x_n\}$ is a Cauchy sequence in E. Indeed, from $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\lim_{n\to\infty} d(x_n,\mathcal{F})$, for any given $\epsilon > 0$, there exists a positive n_0 such that for all $n \geq n_0$,

$$d(x_n,\mathcal{F}) < \frac{\epsilon}{12}, \ \sum_{n=n_0}^{\infty}\delta_n < \frac{\epsilon}{3M} \text{ and } \sum_{n=n_0}^{\infty}\lambda_n < \frac{\epsilon}{3}. \tag{3.10}$$

By (3.10), It is well known that there exists $x_0^* \in \mathcal{F}$ such that

$$\|x_{n_0} - x_0^*\| < \frac{\epsilon}{6}. \tag{3.11}$$

By combining (3.7), (3.10) and (3.11), for any positive integer $m \geq 1$

$$\begin{aligned}
\|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - x_0^*\| + \|x_{n_0} - x_0^*\| \\
&\leq (\|x_{n_0} - x_0^*\| + M\sum_{i=n_0}^{n_0+m-1}\delta_i + \sum_{i=n_0}^{n_0+m-1}\lambda_i) + \|x_{n_0} - x_0^*\| \\
&= 2\|x_{n_0} - x_0^*\| + M\sum_{i=n_0}^{n_0+m-1}\delta_i + \sum_{i=n_0}^{n_0+m-1}\lambda_i \\
&< \frac{\epsilon}{3} + M(\frac{\epsilon}{3M}) + \frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{3.12}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, thus $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = x^*$. Then $x^* \in K$, because K is a closed subset of E. Since the set of fixed points of nonself-asymptotically nonexpansive mappings is closed, we get

$$\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0 \text{ and } \lim_{n\to\infty} d(x_n,x^*) = 0 \tag{3.13}$$

Therefore,

$$d(x^*,\mathcal{F}) = 0 \tag{3.14}$$

and hence we have $x^* \in \mathcal{F}$. This completes our proof. ∎

Theorem 3.3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E(i = 1,2,3)$ be three nonself-asymptotically nonexpansive mappings with respect to P with sequences $\{t_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ and suppose $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and closed. For an arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.9) satisfying the following conditions:*

(*i*) $\liminf_{n\to\infty} \alpha_{n1} > 0$, $\liminf_{n\to\infty} \alpha_{n2} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n3} \leq \limsup_{n\to\infty}(\alpha_{n3} + \gamma_{n3}) < 1$, *then* $\lim_{n\to\infty} \|x_n - (PT_3)^n x_n\| = 0$.

(*ii*) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n2} \leq \limsup_{n\to\infty}(\alpha_{n2} + \beta_{n2} + \gamma_{n2}) < 1$, *then* $\lim_{n\to\infty} \|x_n - (PT_2)^n y_n\| = 0$.

(*iii*) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n1} \leq \limsup_{n\to\infty}(\alpha_{n1} + \beta_{n1} + \gamma_{n1}) < 1$, *then* $\lim_{n\to\infty} \|x_n - (PT_1)^n y_{n+1}\| = 0$.

Proof. By Theorem 3.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for $x^* \in \mathcal{F}$. Thus, it follows that $\{x_n - x^*\}$, $\{y_n - x^*\}$, $\{y_{n+1} - x^*\}$ are all bounded. Since T_i is nonself-asymptotically nonexpansive mappings, we can prove the sequences $\{(PT_i)^n y_n - x^*\}_{i=1}^{2}$, $\{(PT_i)^n x_n - x^*\}_{i=2}^{3}$, $\{(PT_i)^n y_{n+1} - x^*\}$ are all bounded. We may assume that such sequences belong to B_R where $R > 0$. By using (2.9) and Lemma 2.5 we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(1-\alpha_{n3}-\gamma_{n3})x_n + \alpha_{n3}(PT_3)^n x_n + \gamma_{n3}\mu_{n3} - x^*\|^2 \\
&= \|(1-\alpha_{n3}-\gamma_{n3})(x_n - x^*) + \alpha_{n3}((PT_3)^n x_n - x^*) + \gamma_{n3}(\mu_{n3} - x^*)\|^2 \\
&\leq (1-\alpha_{n3}-\gamma_{n3})\|x_n - x^*\|^2 + \alpha_{n3}\|(PT_3)^n x_n - x^*\|^2 + \gamma_{n3}\|\mu_{n3} - x^*\|^2 \\
&\quad - \alpha_{n3}(1-\alpha_{n3}-\gamma_{n3})g(\|x_n - (PT_3)^n x_n\|) \\
&\leq \|x_n - x^*\|^2 + Rt_n + \gamma_{n3}M_{x^*}^2 \\
&\quad - \alpha_{n3}(1-\alpha_{n3}-\gamma_{n3})g(\|x_n - (PT_3)^n x_n\|)
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\|y_{n+1} - x^*\|^2 &= \|(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})x_n + \alpha_{n2}(PT_2)^n y_n \\
&\quad + \beta_{n2}(PT_3)^n x_n + \gamma_{n2}\mu_{n2} - x^*\|^2 \\
&= \|(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})(x_n - x^*) + \alpha_{n2}((PT_2)^n y_n - x^*) \\
&\quad + \beta_{n2}((PT_3)^n x_n - x^*) + \gamma_{n2}(\mu_{n2} - x^*)\|^2 \\
&\leq (1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})\|x_n - x^*\|^2 + \alpha_{n2}\|(PT_2)^n y_n - x^*\|^2 \\
&\quad + \beta_{n2}\|(PT_3)^n x_n - x^*\|^2 + \gamma_{n2}\|\mu_{n2} - x^*\|^2 \\
&\quad - \alpha_{n2}(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})g(\|x_n - (PT_2)^n y_n\|) \\
&\leq (1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})\|x_n - x^*\|^2 \\
&\quad + \alpha_{n2}t_n^2\|y_n - x^*\|^2 + \beta_{n2}t_n^2\|x_n - x^*\|^2 + \gamma_{n2}M_{x^*}^2 \\
&\quad - \alpha_{n2}(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})g(\|x_n - (PT_2)^n y_n\|) \\
&\leq \|x_n - x^*\|^2 + 2Rt_n + (\gamma_{n2}+\gamma_{n3})M_{x^*}^2 \\
&\quad - \alpha_{n2}\alpha_{n3}(1-\alpha_{n3}-\gamma_{n3})g(\|x_n - (PT_3)^n x_n\|) \\
&\quad - \alpha_{n2}(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})g(\|x_n - (PT_2)^n y_n\|)
\end{aligned} \tag{3.16}$$

Similarly, using (2.9), Lemma 2.5, (3.15) and (3.16) we have

$$\|x_{n+1} - x^*\|^2 = \|(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})x_n + \alpha_{n1}(PT_1)^n y_{n+1}$$

$$
\begin{aligned}
&+\beta_{n1}(PT_2)^n y_n+\gamma_{n1}\mu_{n1}-x^*\|^2\\
&=\|(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})(x_n-x^*)+\alpha_{n1}((PT_1)^n y_{n+1}-x^*)\\
&\quad+\beta_{n1}((PT_2)^n y_n-x^*)+\gamma_{n1}(\mu_{n1}-x^*)\|^2\\
&\le(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})\|x_n-x^*\|^2+\alpha_{n1}\|(PT_1)^n y_{n+1}-x^*\|^2\\
&\quad+\beta_{n1}\|(PT_2)^n y_n-x^*\|^2+\gamma_{n1}\|\mu_{n1}-x^*\|^2\\
&\quad-\alpha_{n1}(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})g(\|x_n-(PT_1)^n y_{n+1}\|)\\
&\le(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})\|x_n-x^*\|^2\\
&\quad+\alpha_{n1}t_n^2\|y_{n+1}-x^*\|^2+\beta_{n1}t_n^2\|y_n-x^*\|^2+\gamma_{n1}M_{x^*}^2\\
&\quad-\alpha_{n1}(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})g(\|x_n-(PT_1)^n y_{n+1}\|)\\
&\le\|x_n-x^*\|^2+3Rt_n+(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)M_{x^*}^2\\
&\quad-\alpha_{n1}\alpha_{n2}\alpha_{n3}(1-\alpha_{n3}-\gamma_{n3})g(\|x_n-(PT_3)^n x_n\|)\\
&\quad-\alpha_{n1}\alpha_{n2}(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})g(\|x_n-(PT_2)^n y_n\|)\\
&\quad-\alpha_{n1}(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})g(\|x_n-(PT_1)^n y_{n+1}\|).
\end{aligned}
\tag{3.17}
$$

From (3.17), we obtain the following three important inequalities:

$$
\begin{aligned}
&\alpha_{n1}\alpha_{n2}\alpha_{n3}(1-\alpha_{n3}-\gamma_{n3})g(\|x_n-(PT_3)^n x_n\|)\\
&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+3Rt_n+(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)M_{x^*}^2
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&\alpha_{n1}\alpha_{n2}(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})g(\|x_n-(PT_2)^n y_n\|)\\
&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+3Rt_n+(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)M_{x^*}^2
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
&\alpha_{n1}(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})g(\|x_n-(PT_1)^n y_{n+1}\|)\\
&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+3Rt_n+(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)M_{x^*}^2.
\end{aligned}
\tag{3.20}
$$

If $\liminf_{n\to\infty}\alpha_{n1}>0$, $\liminf_{n\to\infty}\alpha_{n2}>0$ and $0<\liminf_{n\to\infty}\alpha_{n3}<\limsup_{n\to\infty}(\alpha_{n3}+\gamma_{n3})<1$, then there exist a positive integer n_0 and $\tau_1,\tau_2,\tau_3,\tau_4\in(0,1)$ such that $0<\tau_1<\alpha_{n1}$, $0<\tau_2<\alpha_{n2}$, $0<\tau_3<\alpha_{n3}$ and $\alpha_{n3}+\gamma_{n3}<\tau_4<1$, for $n_0>n$. Using above inequalities, we get from (3.18) that

$$
\begin{aligned}
\tau_1\tau_2\tau_3(1-\tau_4)g(\|x_n-(PT_3)^n x_n\|)\le{}&\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+3Rt_n\\
&+(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)M_{x^*}^2.
\end{aligned}
\tag{3.21}
$$

By (3.21) for all $m > n_0$, we derive

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|x_n - (PT_3)^n x_n\|) &\le \frac{1}{\tau_1\tau_2\tau_3(1-\tau_4)}\Big(\sum_{n=n_0}^{m}(\|x_n - x^*\|^2 - \|x_{n+1}-x^*\|^2)\\ &\quad + 3R\sum_{n=n_0}^{m} t_n + M_{x^*}^2\sum_{n=n_0}^{m}(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)\Big)\\ &\le \frac{1}{\tau_1\tau_2\tau_3(1-\tau_4)}\Big(\|x_{n_0}-x^*\|^2\\ &\quad + 3R\sum_{n=n_0}^{m} t_n + M_{x^*}^2\sum_{n=n_0}^{m}(\gamma_{n1}+\gamma_{n2}t_n^2+\gamma_{n3}t_n^2+\gamma_{n3}t_n^4)\Big).\end{aligned} \tag{3.22}$$

Since $\sum_{n=n_0}^{m} t_n < \infty$ and $\sum_{n=n_0}^{m} \gamma_{ni} < \infty$ for $i = 1, 2, 3$, by letting $m \to \infty$ in inequality (3.22), we get that $\sum_{n=n_0}^{m} g(\|x_n - (PT_3)^n x_n\|) < \infty$ and therefore $\lim_{n\to\infty} g(\|x_n - (PT_3)^n x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it implies that

$$\lim_{n\to\infty} \|x_n - (PT_3)^n x_n\| = 0. \tag{3.23}$$

Hence (i) is proved.

(ii) If $\liminf_{n\to\infty} \alpha_{n1} > 0$ and $0 < \liminf_{n\to\infty} \alpha_{n2} \le \limsup_{n\to\infty}(\alpha_{n2} + \beta_{n2} + \gamma_{n2}) < 1$, then by using a similar method, together with inequality (3.19), it can be shown that

$$\lim_{n\to\infty} \|x_n - (PT_2)^n y_n\| = 0. \tag{3.24}$$

(iii) If $\liminf_{n\to\infty} \alpha_{n1} > 0$ and $0 < \liminf_{n\to\infty} \alpha_{n1} \le \limsup_{n\to\infty}(\alpha_{n1} + \beta_{n1} + \gamma_{n1}) < 1$, then by using a similar method, together with inequality (3.20), it can be shown that

$$\lim_{n\to\infty} \|x_n - (PT_1)^n y_{n+1}\| = 0. \tag{3.25}$$

This completes our proof. ∎

Theorem 3.4. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_i : K \to E(i = 1, 2, 3)$ be three nonself-asymptotically nonexpansive mappings with respect to P with sequences $\{t_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ and suppose $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and closed. For an arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.9) satisfying the following conditions:*

(i) $\liminf_{n\to\infty} \alpha_{n1} > 0$, $\liminf_{n\to\infty} \alpha_{n2} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n3} \le \limsup_{n\to\infty}(\alpha_{n3} + \gamma_{n3}) < 1$,

(ii) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{nj} \le \limsup_{n\to\infty}(\alpha_{nj} + \beta_{nj} + \gamma_{nj}) < 1$, *for* $j = 1, 2$.

Then $\lim_{n\to\infty} (\|x_n - (PT_i)x_n\|) = 0$, *for* $j = 1, 2, 3$.

Proof. On the other hand note that

$$\begin{aligned}\|y_n - x_n\| &= \|(1-\alpha_{n3}-\gamma_{n3})x_n + \alpha_{n3}(PT_3)^n x_n + \gamma_{n3}\mu_{n3} - x_n\| \\ &\le \alpha_{n3}\|x_n - (PT_3)^n\| + \gamma_{n3}\|\mu_{n3} - x_n\| \to 0 \ (n\to\infty).\end{aligned} \tag{3.26}$$

Since T_2 nonself-asymptotically nonexpansive mapping, from (3.24) and (3.26), we get

$$\begin{aligned}\|x_n - (PT_2)^n x_n\| &\le \|x_n - (PT_2)^n y_n\| + \|(PT_2)^n y_n - (PT_2)^n x_n\| \\ &\le \|x_n - (PT_2)^n y_n\| + t_n\|y_n - x_n\| \to 0 \ (n\to\infty).\end{aligned} \tag{3.27}$$

It follows from (3.23) and (3.24) that

$$\begin{aligned}\|y_{n+1} - x_n\| &= \|(1-\alpha_{n2}-\beta_{n2}-\gamma_{n2})x_n + \alpha_{n2}(PT_2)^n y_n \\ &\quad + \beta_{n2}(PT_3)^n x_n + \gamma_{n2}\mu_{n2} - x_n\| \\ &\le \alpha_{n2}\|x_n - (PT_2)^n y_n\| + \beta_{n2}\|x_n - (PT_3)^n x_n\| + \gamma_{n2}\|\mu_{n2} - x_n\| \\ &\quad \to 0 \ (n\to\infty).\end{aligned} \tag{3.28}$$

From (3.25) and (3.28), we have

$$\begin{aligned}\|x_n - (PT_1)^n x_n\| &\le \|x_n - (PT_1)^n y_{n+1}\| + \|(PT_1)^n y_{n+1} - (PT_1)^n x_n\| \\ &\le \|x_n - (PT_1)^n y_{n+1}\| + t_n\|y_{n+1} - x_n\| \to 0 \ (n\to\infty).\end{aligned} \tag{3.29}$$

Hence, by (2.9), (3.24) and (3.25), we get that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\alpha_{n1}-\beta_{n1}-\gamma_{n1})x_n + \alpha_{n1}(PT_1)^n y_{n+1} \\ &\quad + \beta_{n1}(PT_2)^n x_n + \gamma_{n1}\mu_{n1} - x_n\| \\ &\le \alpha_{n1}\|x_n - (PT_1)^n y_{n+1}\| + \beta_{n1}\|x_n - (PT_2)^n y_n\| + \gamma_{n1}\|\mu_{n1} - x_n\| \\ &\quad \to 0 \ (n\to\infty).\end{aligned} \tag{3.30}$$

Since an asymptotically nonexpansive mapping with respect to P must be uniformly Lipschitzian with respect to P, then we have

$$\begin{aligned}\|x_{n+1} - (PT_i)x_{n+1}\| &\le \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + \|(PT_i)^{n+1}x_{n+1} - (PT_i)x_{n+1}\| \\ &\le \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + L\|(PT_i)x_{n+1} - (PT_i)^n x_{n+1}\| \\ &\le \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + L\|(x_{n+1} - x_n) \\ &\quad + (x_n - (PT_i)^n x_n) + ((PT_i)^n x_n - (PT_i)x_{n+1})\| \\ &\le \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + L\|\|x_{n+1} - x_n\| \\ &\quad + L\|x_n - (PT_i)^n x_n\| + L\|(PT_i)^n x_n - (PT_i)x_{n+1}\| \\ &\le \|x_{n+1} - (PT_i)^{n+1}x_{n+1}\| + L\|x_{n+1} - x_n\| \\ &\quad + L\|x_n - (PT_i)^n x_n\| + L^2\|x_n - x_{n+1}\|.\end{aligned} \tag{3.31}$$

This with (3.23), (3.27), (3.29) and (3.30) implies that $\lim_{n\to\infty}\|x_n - (PT_i)x_n\| = 0$ for $i = 1, 2, 3$. This completes our proof. ∎

Theorem 3.5. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E which P as a sunny nonexpansive retraction. Let $T_i : K \to E(i = 1, 2, 3)$ be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with sequences $\{t_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ and suppose $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and closed. For an arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.9) satisfying the following conditions:*

(*i*) $\liminf_{n\to\infty} \alpha_{n1} > 0$, $\liminf_{n\to\infty} \alpha_{n2} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n3} \leq \limsup_{n\to\infty}(\alpha_{n3} + \gamma_{n3}) < 1$,

(*ii*) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{nj} \leq \limsup_{n\to\infty}(\alpha_{nj} + \beta_{nj} + \gamma_{nj}) < 1$, *for* $j = 1, 2$.

If $\{T_1, T_2, T_3\}$ satisfies Condition (B) with respect to sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3\}$.

Proof. By Theorem 3.4, we have $\lim_{n\to\infty} \|x_n - (PT_i)x_n\| = 0$, for $i = 1, 2, 3$. Since $\{T_1, T_2, T_3\}$ satisfy Condition (B) with respect to sequence $\{x_n\}$, we get that

$$\max_{1\leq i\leq 3}\{\|x_n - (PT_i)x_n\|\} \geq f(d(x, \mathcal{F})). \tag{3.32}$$

Therefore, $\lim_{n\to\infty} d(x, \mathcal{F}) = 0$. Since f is a nondecreasing function and $f(0) = 0$, hence $\lim_{n\to\infty} d(x, \mathcal{F}) = 0$. Now, applying the theorem 3.2, we obtain the result. This completes our proof. ∎

Remark 3.6. [8]. Noting that $x_n = Px_n$ for all $n \geq 1$, we have $\|x_n - (PT)x_n\| \leq \|x_n - Tx_n\|$ for all $n \geq 1$. Therefore, the Condition (B) is weaker than the Condition (A).

Corollary 3.7. *Let E be a real smooth and uniformly convex Banach space and K be a nonempty closed convex subset of E which P as a sunny nonexpansive retraction. Let $T_i : K \to E(i = 1, 2, 3)$ be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with sequences $\{t_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ and suppose $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and closed. For an arbitrary $x_1 \in K$, suppose that $\{x_n\}$ is the sequence defined by (2.9) satisfying the following conditions:*

(*i*) $\liminf_{n\to\infty} \alpha_{n1} > 0$, $\liminf_{n\to\infty} \alpha_{n2} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n3} \leq \limsup_{n\to\infty}(\alpha_{n3} + \gamma_{n3}) < 1$,

(*ii*) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{nj} \leq \limsup_{n\to\infty}(\alpha_{nj} + \beta_{nj} + \gamma_{nj}) < 1$, *for* $j = 1, 2$.

If one of $\{T_1, T_2, T_3\}$ is completely continuous, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3\}$.

Proof. Since one of $\{T_1, T_2, T_3\}$ is demi-compact, so is one of $\{PT_1, PT_2, PT_3\}$. By continuity of P, T_1, T_2 and T_3, so that PT_1, PT_2, PT_3 are all continuous. It is well known that every continuous and demi-compact mapping must satisfy Condition (B). Hence, the conclusion of the corollary follows from Theorem 3.5. This completes our proof. ∎

Theorem 3.8. *Let E be a real smooth and uniformly convex Banach space and K be a nonempty closed convex subset of E satisfying Opial's condition which P as a sunny nonexpansive retraction. Let $T_i : K \to E(i = 1, 2, 3)$ be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to P with sequences $\{t_n^{(i)}\}$ such that*

$\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ *and suppose* $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ *and closed. For an arbitrary* $x_1 \in K$*, suppose that* $\{x_n\}$ *is the sequence defined by (2.9) satisfying the following conditions:*

(*i*) $\liminf_{n\to\infty} \alpha_{n1} > 0$, $\liminf_{n\to\infty} \alpha_{n2} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n3} \leq \limsup_{n\to\infty}(\alpha_{n3} + \gamma_{n3}) < 1$,

(*ii*) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{nj} \leq \limsup_{n\to\infty}(\alpha_{nj} + \beta_{nj} + \gamma_{nj}) < 1$, *for* $j = 1, 2$.

Then $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, T_3\}$.

Proof. For any $x^* \in \mathcal{F}$. By Theorem 3.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists, and $\{x_n\}$ is bounded. We now show that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F}$. Assume that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to x_1^* and x_2^*, respectively. By Theorem 3.4, we have $\lim_{n\to\infty} \|x_n - (PT_i)x_n\| = 0$, for $i = 1, 2, 3$. Lemma 2.7 implies that $(I - PT_i)x_1^* = 0$, that is, $(PT_i)x_1^* = x_1^*$. Similiarly, we obtain that $(PT_i)x_2^* = x_2^*$. Also Lemma 2.6 guarantees that $x_1^*, x_2^* \in \mathcal{F}$. Next, we prove the uniqueness. For this, suppose that $x_1^* = x_2^*$. Then, by Opial's condition, we get that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - x_1^*\| &= \lim_{k\to\infty} \|x_{n_k} - x_1^*\| < \lim_{k\to\infty} \|x_{n_k} - x_2^*\| \\ &= \lim_{n\to\infty} \|x_n - x_2^*\| = \lim_{j\to\infty} \|x_{n_j} - x_2^*\| \\ &< \lim_{j\to\infty} \|x_{n_j} - x_1^*\| = \lim_{n\to\infty} \|x_n - x_1^*\|, \end{aligned} \tag{3.33}$$

which is a contradiction. Thus $\{x_n\}$ converges weakly to a point of $\mathcal{F}$. This completes our proof. ∎

Corollary 3.9. *Let* E *be a real smooth and uniformly convex Banach space and* K *be a nonempty closed convex subset of* E *satisfying Opial's condition which* P *as a sunny nonexpansive retraction. Let* $T_i : K \to E (i = 1, 2, 3)$ *be three weakly inward and nonself-asymptotically nonexpansive mappings with respect to* P *with sequences* $\{t_n^{(i)}\}$ *such that* $\sum_{n=1}^{\infty} t_n^{(i)} < \infty$ *and suppose* $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ *and closed. For an arbitrary* $x_1 \in K$*, suppose that* $\{x_n\}$ *is the sequence defined by (2.9) satisfying the following conditions:*

(*i*) $\liminf_{n\to\infty} \alpha_{n1} > 0$, $\liminf_{n\to\infty} \alpha_{n2} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{n3} \leq \limsup_{n\to\infty}(\alpha_{n3} + \gamma_{n3}) < 1$,

(*ii*) $\liminf_{n\to\infty} \alpha_{n1} > 0$ *and* $0 < \liminf_{n\to\infty} \alpha_{nj} \leq \limsup_{n\to\infty}(\alpha_{nj} + \beta_{nj} + \gamma_{nj}) < 1$, *for* $j = 1, 2$.

Then $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_1, T_2, T_3\}$.

Remark 3.10. All the above theorems, the iterative sequence (2.11), (2.12) and (2.14) can be replaced by the three step iterative process (2.9), then the results of this paper still hold.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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