Bangmod Int. **J.** **M**ath. **&** **C**omp. **S**ci.
Vol. 1, No. 2, 2015; Pages 221 - 232
http://bangmod-jmcs.kmutt.ac.th/
ISSN : 2408-154X

SOME TIME-OPTIMAL CONTROL PROBLEMS FOR $n \times n$ CO-OPERATIVE HYPERBOLIC SYSTEMS WITH CONTROL IN INITIAL CONDITIONS

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Abstract In this paper, various time-optimal control problems for $n \times n$ co-operative hyperbolic linear system involving Laplace operator is considered. The controls act in initial conditions. For each problem, the optimal controls are characterized in terms of an adjoint system and shown to be unique and bang-bang.

MSC: 35K35, 49K20, 49J20, 93C20.

Keywords: Time-optimal control problems, hyperbolic system, co-operative systems.

Submission date: 9 April 2015 / Acceptance date: 19 April 2015 /Available online 20 April 2015

1. Introduction

Optimal control problems with controls in initial conditions have been appeared in some few papers [3, 10, 12]. In [12], [3], optimal control problems for parabolic and hyperbolic equations, with control via initial conditions have been studied and in [10] optimal control of infinite order hyperbolic equation with control via initial conditions have been considered.

Time-optimal control of distributed parameter systems governed by a system of hyperbolic equations is of special importance for the active control of structural systems for which the equations of motion are generally expressed by hyperbolic differential equations. A typical application of a hyperbolic equation is the vibrating system. Time-optimal control of distributed parameter systems governed by a system of hyperbolic equations have been studied in many papers, from them, we mention only two papers [6],[11] in which time optimal distributed control problems of vibrating systems has been studied. In our

Published by **T**heoretical **a**nd **C**omputational **S**cience Center **(TaCS)**,
King Mongkut's University of Technology Thonburi **(KMUTT)**

papers [4],[15] the results in [6] and [11] have been extended to the time optimal control problems for systems governed by $n \times n$ hyperbolic systems, involving Laplace operator with distributed and boundary controls. In our paper [16], the results have been extended to $n \times n$ co-operative parabolic systems with controls in initial conditions.

In this paper, we will consider various time-optimal control problems for $n \times n$ co-operative linear hyperbolic system involving Laplace operator as well as controls acting in initial conditions (here and everywhere below the vectors are denoted by bold letters.):

Consider the following $n \times n$ co-operative linear hyperbolic system involving Laplace operator

$$\left.\begin{array}{rl} \dfrac{\partial^2 y_i}{\partial t^2}(x,t) - (A(t)\mathbf{y})_i = f_i(x,t) & \text{in } Q = \Omega \times]0,T[, \\ y_i(x,0) = u_i(x) & \text{in } \Omega, \\ \dfrac{\partial y_i}{\partial t}(x,0) = v_i(x) & \text{in } \Omega, \\ y_i(x,t) = 0, & \text{on } \Sigma = \Gamma \times]0,T[, \end{array}\right\} \tag{1.1}$$

where f_i, u_i, v_i are given functions, $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a bounded open domain with a smooth boundary Γ and $\{A(t) : t \in]0,T[\}$ is a family of $n \times n$ continuous matrix operators;

$$A(t)y = \begin{pmatrix} \Delta + a_1 & a_{12} & \dots & a_{1n} \\ a_{21} & \Delta + a_2 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

with co-operative coefficient functions a_i, a_{ij} satisfying the following conditions;

$$\left.\begin{array}{l} a_i, a_{ij} \text{ are positive functions in } L^\infty(Q), \\ a_{ij} = a_{ji} \text{ (symmetry conditions)}, \\ a_{ij}(x,t) \le \sqrt{a_i(x,t)a_j(x,t)}. \end{array}\right\} \tag{1.2}$$

These problems are steering the initial vector state $\mathbf{y}(0) = \mathbf{u}$ or $\mathbf{y}(0) = \mathbf{v}$ for system (1.1) with a position vector control $\mathbf{u} = (u_1, u_2, ..., u_n)$ or a velocity vector control $\mathbf{v} = (v_1, v_2, ..., v_n)$, belonging to a given control set U_ϵ^n so that an observation $\mathbf{y}(t)$ or $\mathbf{y}'(t)$ hitting a given target set K_ε^n in a minimum time,

$$\begin{array}{l} U_\epsilon^n = \{\phi = (\phi_1, \phi_2, ..., \phi_n) \in (L^2(\Omega))^n : \ \|\phi_i\|_{L^2(\Omega)} \le \epsilon\} \\ K_\varepsilon^n = \{z = (z_1, z_2, ..., z_n) \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} \le \varepsilon\}, \end{array} \tag{1.3}$$

and ϵ, $\varepsilon > 0$ and $z_{id} \in L^2(\Omega)$ are given.

First, we establish the well posedness of the system (1.1) under conditions on the coefficients stated by the principal eigenvalue of the Laplace eigenvalue problem. Then, we formulate various time optimal initial control problems and in each problem, we derive the necessary and sufficient conditions which the optimal controls must satisfy in terms of the adjoint.

2. $n \times n$ Co-operative hyperbolic systems

Let $H_0^1(\Omega)$, be the usual Sobolev space of order one which consists of all $\phi \in L^2(\Omega)$ whose distributional derivatives $\frac{\partial \phi}{\partial x_i} \in L^2(\Omega)$ and $\phi_\Gamma = 0$ with the scalar product norm

$$\langle y, \phi \rangle_{H_0^1(\Omega)} = \langle y, \phi \rangle_{L^2(\Omega)} + \langle \nabla y, \nabla \phi \rangle_{L^2(\Omega)}, \quad \text{where } \nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}.$$

We have the following dense embedding chain [2]

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H_0^{-1}(\Omega))^n,$$

where $H_0^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$.

For $\mathbf{y} = (y_i)_{i=1}^n$, $\phi = (\phi_i)_{i=1}^n \in (H_0^1(\Omega))^n$ and $t \in]0, T[$, let us define a family of continues bilinear forms

$$\pi(t; \cdot, \cdot) : (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \to \mathbb{R} \quad \text{by}$$

$$\begin{aligned} \pi(t; \mathbf{y}, \phi) &= \sum_{i=1}^{n} \int_\Omega \left[(\nabla y_i)(\nabla \phi_i) - a_i(x,t) y_i \phi_i \right] dx - 2 \sum_{i>j}^{n} \int_\Omega a_{ij}(x,t) y_j \phi_i dx \\ &= \sum_{i=1}^{n} \int_\Omega \left[(-\Delta y_i) - a_i(x,t) y_i \right] \phi_i dx - 2 \sum_{i>j}^{n} \int_\Omega a_{ij}(x,t) y_j \phi_i dx \qquad (2.1) \\ &= \sum_{i=1}^{n} \langle -(A(t)\mathbf{y})_i, \phi \rangle_{L^2(\Omega)} . \end{aligned}$$

Lemma 1. *If Ω is a regular bounded domain in $\mathbb{R}^{\mathbb{N}}$, with a boundary Γ, and if m is positive on Ω and smooth enough (in particular $m \in L^\infty(\Omega)$), then the eigenvalue problem;*

$$\left. \begin{aligned} -\Delta y = \lambda m(x) y \qquad & \text{in } \Omega, \\ y = 0 \qquad & \text{on } \Gamma \end{aligned} \right\}$$

possesses an infinite sequence of positive eigenvalues;

$$0 < \lambda_1(m) < \lambda_2(m) \leq \cdots \lambda_k(m) \cdots ; \lambda_k(m) \to \infty, \quad \text{as } k \to \infty.$$

Moreover $\lambda_1(m)$ is simple, its associate eigenfunction e_m is positive, and $\lambda_1(m)$ is characterized as

$$\lambda_1(m) \int_\Omega m y^2 dx \leq \int_\Omega |\nabla y|^2 dx. \qquad (2.2)$$

Proof. See [9]. ∎

Now, let

$$\lambda_1(a_i) \geq n, \quad i = 1, 2, ..., n. \qquad (2.3)$$

Lemma 2. *If (1.2) and (2.3) be held then, the bilinear form (2.1) satisfies the Gårding inequality*

$$\pi(t; \mathbf{y}, \mathbf{y}) + c_0 \|y\|^2_{(L^2(\Omega))^n} \geq c_1 \|y\|^2_{(H_0^1(\Omega))^n}, \quad c_0, \ c_1 > 0.$$

Proof. In fact,

$$\pi(t;\mathbf{y},\mathbf{y}) = \sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2 - a_i(x,t)y_i^2\right]dx - \sum_{i,j=1}^{n}\int_{\Omega}a_{ij}(x,t)y_iy_jdx$$
$$\geq \sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2 - a_i(x,t)y_i^2\right]dx - 2\sum_{i>j}^{n}\int_{\Omega}\sqrt{a_i(x,t)a_j(x,t)}y_iy_jdx$$

By Cauchy Schwartz inequality and (2.2), we obtain

$$\pi(t;\mathbf{y},\mathbf{y}) \geq \sum_{i=1}^{n}\left(1-\frac{1}{\lambda_1(a_i)}\right)\int_{\Omega}|\nabla y_i|^2dx$$
$$-2\sum_{i>j}^{n}\int_{\Omega}\left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}}\right)\left(\int_{\Omega}|\nabla y_i|^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega}|\nabla y_j|^2dx\right)^{\frac{1}{2}}$$
$$\geq \sum_{i=1}^{n}\left(\frac{\lambda_1(a_i)-n}{\lambda_1(a_i)}\right)\int_{\Omega}|\nabla y_i|^2dx$$

From (2.3) we have

$$\pi(t;\mathbf{y},\mathbf{y}) \geq \alpha\left[\sum_{i=1}^{n}\int_{\Omega}|\nabla y_i|^2dx\right] \quad \alpha > 0.$$

Adding $\|\mathbf{y}\|_{(L^2(\Omega))^n}$ to two both sides, we have the desired result. ∎

For optimal control problems it is of importance to consider the cases where the control v_i or u_i belongs to $L^2(\Omega)$. For these cases, we have the following results (by apply Theorem 1.1 and Remark.1.3 Chapter 4 in [12] with $V = (H_0^1(\Omega))^n$ and $H = (L^2(\Omega))^n$):

Theorem 1. *Let* (1.2) *and* (2.3) *be hold and let* $f_i, v_i\ u_i$ *be given as*

$$f_i \in L^2(Q), \quad u_i \in H^1(\Omega), \quad v_i \in L^2(\Omega).$$

Then there exists a unique solution $\mathbf{y} \in \left\{\mathbf{y} : \mathbf{y} \in L^2(0,T;(H_0^1(\Omega))^n), \quad \frac{\partial \mathbf{y}}{\partial t} \in (L^2(Q))^n\right\}$ *satisfying the Dirichlet problem* (1.1). *Moreover* $\mathbf{y}$ *is continuous from* $[0,T]$ *to* $(H_0^1(\Omega))^n$ *and* $\frac{\partial \mathbf{y}}{\partial t}$ *is continuous from* $[0,T]$ *to* $(L^2(\Omega))^n$.

By transposition (see [12], [14]), we deduce the following:

Theorem 2. *(Transposition theorem) Let* (1.2) *and* (2.3) *be held and let* $f_i, v_i\ u_i$ *be given as*

$$f_i \in L^1(0,T;H_0^{-1}(\Omega)), \quad u_i \in L^2(\Omega), \quad v_i \in H^{-1}(\Omega).$$

Then there exists a unique solution $\mathbf{y} \in (L^2(Q))^n$ *satisfying the Dirichlet problem* (1.1). *Moreover* $\mathbf{y}$ *is continuous from* $[0,T]$ *to* $(L^2(\Omega))^n$ *and* $\frac{\partial \mathbf{y}}{\partial t}$ *is continuous from* $[0,T]$ *to* $(H_0^{-1}(\Omega))^n$.

In the next sections, we will denote by $\mathbf{y}(t;\mathbf{v})$ to the unique solution of (1.1), at time t corresponding to a given control $\mathbf{v} \in U_\epsilon^n$ and a given functions f_i, u_i, v_i satisfying the hypothesis of Theorem 1. Similarly, we will denote by $\mathbf{y}(t;\mathbf{u})$ to the unique solution of (1.1), at time t corresponding to a given control $\mathbf{u} \in U_\epsilon^n$ and a given functions f_i, u_i, v_i satisfying the hypothesis of Theorem 2. Occasionally, we write $\mathbf{y}(x,t;\mathbf{v})$ or $\mathbf{y}(x,t;\mathbf{u})$ when the explicit dependence on x is required.

3. Velocity control-position observation problem

In this section, we consider the following first time-optimal control problem with a control $\mathbf{v}$ acting in velocity initial condition and position observation $\mathbf{y}(x,t;\mathbf{v})$:

$$(TOP1): \qquad \min\{t: \ \mathbf{y}(x,t;\mathbf{v}) \in K^n_\varepsilon, \ \ \mathbf{v} \in U^n_\epsilon\}.$$

In order for this problem to be well-posed, we assume that the system whose state is given by (1.1) is controllable ([13], [5]),

$$\text{i.e., there exist a } \tau \in]0,T] \text{ and } \mathbf{v} \in U^n_\epsilon \text{ with } \mathbf{y}(\tau;\mathbf{v}) \in K^n_\varepsilon. \tag{3.1}$$

Set

$$\tau^0_1 = \inf\{\tau : \mathbf{y}(\tau;\mathbf{v}) \in K^n_\varepsilon \ \text{ fore some } \ \mathbf{v} \in U^n_\epsilon\}. \tag{3.2}$$

The following result holds .

Theorem 3. *If (1.2) and (2.3) are held, then there exists an admissible control $\mathbf{v}^0$ to the problem (TOP1), which steering $\mathbf{y}(t;\mathbf{v}^0)$ to hitting a target set K^n_ε in minimum time τ^0_1 (defined by (3.2)). Moreover*

$$\sum_{i=1}^n \int_\Omega (y_i(\tau^0_1;\mathbf{v}^0) - z_{id}) \left(y_i(\tau^0_1;\mathbf{v}) - y_i(\tau^0_1;\mathbf{v}^0)\right) dx \geq 0 \quad \forall \mathbf{v} \in U^n_\epsilon. \tag{3.3}$$

Proof. Fix x, we can choose $\tau^m \to \tau^0_1$ and admissible controls $\{\mathbf{v}^m\}$ such that

$$\mathbf{y}(\tau^m;\mathbf{v}^m) \in K^n_\varepsilon, \quad m = 1,2,\ldots.$$

Set $\mathbf{y}^m = \mathbf{y}(\mathbf{v}^m)$. Since U^n_ϵ is bounded, we may verify that $\mathbf{y}^m$ (respectively, $\frac{d\mathbf{y}}{dt}$) ranges in a bounded set in $(L^2(0,T;(H^1_0(\Omega))^n))$ (respectively, $(L^2(0,T;(L^2(\Omega))^n) = (L^2(Q))^n$).

We may then extract a subsequence, again denoted by $\{\mathbf{v}^m, \mathbf{y}^m\}$ such that

$$\left.\begin{aligned} &\mathbf{v}^m \to \mathbf{v}^0 \quad \text{weakly in } (L^2(\Omega))^n, \quad v^0 \in U^n_\epsilon,\\ &\mathbf{y}^m \to \mathbf{y} \quad \text{weakly in } L^2\left(0,T;\left(H^1_0(\Omega)\right)^n\right),\\ &\frac{d\mathbf{y}^m}{dt} \to \frac{d\mathbf{y}}{dt} \quad \text{weakly in } (L^2(Q))^n. \end{aligned}\right\} \tag{3.4}$$

We deduce from the equality

$$\frac{d^2\mathbf{y}^m}{dt^2} = f - A(t)\mathbf{y}^m$$

that

$$\frac{d^2\mathbf{y}^m}{dt^2} \to \frac{d^2\mathbf{y}}{dt^2} = f - A(t)\mathbf{y} \quad \text{in } L^2\left(0,T;\left(H^{-1}(\Omega)\right)^n\right),$$

and

$$\left.\begin{aligned} &\mathbf{y}^m(0) \to \mathbf{y}(0) = \mathbf{u} \quad \text{in } (L^2(\Omega))^n,\\ &\frac{d\mathbf{y}^m}{dt}(0) \to \frac{d\mathbf{y}}{dt}(0) = \mathbf{v}^0 \quad \text{in } U^n_\epsilon. \end{aligned}\right\}.$$

But

$$\mathbf{y}(\tau^m;\mathbf{v}^m) - \mathbf{y}(\tau^0_1;\mathbf{v}^0) = \mathbf{y}(\tau^m;\mathbf{v}^m) - \mathbf{y}(\tau^0_1;\mathbf{v}^m) + \mathbf{y}(\tau^0_1;\mathbf{v}^m) - \mathbf{y}(\tau^0_1;\mathbf{v}^0)$$

then, from (3.4) we have

$$\mathbf{y}(\tau_1^0;\mathbf{v}^m) \to \mathbf{y}(\tau_1^0;\mathbf{v}^0) \quad \text{weakly in } (H_0^1(\Omega))^n \tag{3.5}$$

and

$$\begin{aligned}\|\mathbf{y}(\tau^m;\mathbf{v}^m) - \mathbf{y}(\tau_1^0;\mathbf{v}^m)\|_{(L^2(\Omega))^n} &= \left\| \int_{\tau_1^0}^{\tau^m} \frac{d}{dt}\mathbf{y}(t;\mathbf{v}^m)dt \right\|_{(L^2(\Omega))^n} \\ &\le \sqrt{\tau^m - \tau_1^0}\left(\int_{\tau_1^0}^{\tau^m} \left\| \frac{d}{dt}\mathbf{y}(t;\mathbf{v}^m)dt \right\|^2_{(L^2(\Omega))^n} dt \right)^{\frac{1}{2}} \\ &\le c\sqrt{\tau_n - \tau_1^0}.\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) shows that

$$\mathbf{y}(\tau^m;\mathbf{v}^m) - \mathbf{y}(\tau_1^0;\mathbf{v}^0) \to 0 \quad \text{weakly in } \left(L^2(\Omega)\right)^n. \tag{3.7}$$

Similarly, we can verify that

$$\mathbf{y}'(\tau^m;\mathbf{v}^m) - \mathbf{y}'(\tau_1^0;\mathbf{v}^0) \to 0 \quad \text{weakly in } \left(H_0^{-1}(\Omega)\right)^n. \tag{3.8}$$

and so, $\mathbf{y}(\tau_1^0;\mathbf{v}^0) \in K_\varepsilon^n$ as K_ε^n is closed and convex, hence weakly closed. This shows that K_ε^n is reached in time τ_1^0 by the admissible control $\mathbf{v}^0$.

For the second part of the theorem, really, from Theorem 1, the mappings $t \mapsto \mathbf{y}(t;\mathbf{v})$ and $t \mapsto \mathbf{y}'(t;\mathbf{v})$ (from $[0,T] \to (H_0^1(\Omega))^n$ and $[0,T] \to (L^2(\Omega))^n$, respectively) are continuous for each fixed $\mathbf{v}$ and so $\mathbf{y}(\tau_1^0;\mathbf{v}) \notin int\, K_\varepsilon^n$, for any $\mathbf{v} \in U_\epsilon^n$, by the minimality of τ_1^0.

Using Theorem 1 it is easy to verify that the mapping $\mathbf{v} \mapsto \mathbf{y}(\tau_1^0;\mathbf{v})$ (from $(L^2(\Omega))^n \to (L^2(\Omega))^n$) is continuous and linear. Then, the set

$$\mathcal{A}(\tau_1^0) = \{\mathbf{y}(\tau_1^0;\mathbf{v}) \,:\, \mathbf{v} \in U_\epsilon^n\}$$

is the image under a linear mapping of a convex set, hence $\mathcal{A}(\tau_1^0)$ is convex. Thus we have $\mathcal{A}(\tau_1^0) \cap intK_\varepsilon^n = \emptyset$ and $\mathbf{y}(\tau_1^0;\mathbf{v}^0) \in \partial K_\varepsilon^n$ (the boundary of K_ε^n). Since $intK_\varepsilon^n \neq \emptyset$ $(from$ (3.1)), there exists a closed hyperplane separating $A(\tau_1^0)$ and K_ε^n containing $\mathbf{y}(\tau_1^0;\mathbf{v}^0)$, i.e., there is a nonzero $\mathbf{g} \in (L^2(\Omega))^n$ such as

$$\sup_{\mathbf{y}\in A(\tau_1^0)} \left\langle \mathbf{g}, \mathbf{y}(\tau_1^0;\mathbf{v}) \right\rangle_{(L^2(\Omega))^n} \le \left\langle \mathbf{g}, \mathbf{y}(\tau_1^0;\mathbf{v}^0) \right\rangle_{(L^2(\Omega))^n} \le \inf_{\mathbf{y}\in K_\varepsilon^n} \left\langle \mathbf{g}, \mathbf{y}(\tau_1^0;\mathbf{v}) \right\rangle_{(L^2(\Omega))^n}. \tag{3.9}$$

From the second inequality in (15), $\mathbf{g}$ must support the set K_ε^n at $\mathbf{y}(\tau_1^0;\mathbf{v}^0)$ i.e.,

$$\langle \mathbf{g}\,, (\mathbf{y}(\tau_1^0;\mathbf{v}) - \mathbf{y}(\tau_1^0;\mathbf{v}^0)) \rangle_{(L^2(\Omega))^n} \ge 0 \quad \forall \mathbf{v} \in U_\epsilon^n$$

and since $(L^2(\Omega))^n$ is a Hilbert space, $\mathbf{g}$ must be of the form

$$\mathbf{g} = \lambda(\mathbf{y}(\tau_1^0;\mathbf{v}^0) - z_d) \quad \text{for some } \lambda > 0.$$

Dividing the inequality (4.1) by λ gives the desired result. ∎

The above condition (3.3) can be simplified by introducing the following adjoint equation. For each $\mathbf{v}^0 \in U_\epsilon^n$, we define $p(x,t;\mathbf{v}^0)$ as the solution of the following system

$$\left.\begin{array}{ll}\dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{v}^0) - \big(A(t)\mathbf{p}(t;\mathbf{v}^0)\big)_i = 0 & \text{in}\,\Omega\times]0,\tau_1^0[,\\ p_i(x,\tau_1^0;\mathbf{v}^0) = 0 & \text{in}\,\Omega,\\ p_i'(x,\tau_1^0;\mathbf{v}^0) = -(y_i(x,\tau_1^0;\mathbf{v}^0) - z_{id}) & \text{in}\,\Omega,\\ p_i(x,t;\mathbf{v}^0) = 0 & \text{in}\,\Gamma\times]0,\tau_1^0[.\end{array}\right\} \tag{3.10}$$

The existence of a unique solution for the problem (3.10) can be proved using Theorem 1 with an obvious change of variables). We multiply the first equation in (3.10) by $y_i(t;\mathbf{v}) - y_i(t;\mathbf{v}^0)$ and integrate by parts from 0 to τ_1^0, we obtain the following identity:

$$\int_\Omega (y_i(x,\tau_1^0;\mathbf{v}^0) - z_{id})(y_i(x,\tau_1^0;\mathbf{v}) - y_i(x,\tau_1^0;\mathbf{v}^0))dx = \int_\Omega p_i(x,0;\mathbf{v}^0)(\mathbf{v}-\mathbf{v}^0)dx.$$

The inequality (3.3) then becomes

$$\sum_{i=1}^n \int_\Omega p_i(x,0;\mathbf{v}^0)(\mathbf{v}-\mathbf{v}^0)dx \geq 0 \quad \forall \mathbf{v}\in U_\epsilon^n. \tag{3.11}$$

This result can be summarized as;

Theorem 4. *We assume that (1.2) and (2.3) hold. Then there exists the adjoint state*

$$\mathbf{p} \in \left\{\mathbf{p} : \mathbf{p} \in L^2(0,\tau_1^0;(H_0^1(\Omega))^n), \quad \frac{\partial \mathbf{p}}{\partial t} \in (L^2(\Omega))^n)\right\}$$

such that the optimal control $\mathbf{v^0}$ of problem (TOP1) is characterized by (3.10) and (3.11) together with (1.1) (with $v_i = v_i^0$, $i = 1,2,...,n$).

4. Velocity control-velocity observation problem

In this section, we consider the following second time-optimal control problem with a control $\mathbf{v}$ acting in a velocity initial condition and a velocity observation $\mathbf{y}'(x,t;\mathbf{v})$:

$$(TOP2): \qquad \min\{t: \ \mathbf{y}'(x,t;\mathbf{v}) \in K_\varepsilon^n, \ \mathbf{v}\in U_\epsilon^n\}.$$

As in the above section, we assume that the following controllability condition is hold

$$\text{There exist a } \tau \in]0,T] \text{ and } \ \mathbf{v}\in U_\epsilon^n \text{ with } \mathbf{y}'(\tau;\mathbf{v}) \in K_\varepsilon^n \tag{4.1}$$

and if we set

$$\tau_2^0 = \inf\{\tau : \mathbf{y}'(\tau;\mathbf{v}) \in K_\varepsilon^n \ \text{ fore some } \ \mathbf{v}\in U_\epsilon^n\}, \tag{4.2}$$

then similarly to (TOP1) we can prove the following theorem;

Theorem 5. *If (1.2) and (2.3) are held, then there exist an admissible control $\mathbf{v}^0$ to the problem (TOP2), which steering $\mathbf{y}'(t;\mathbf{v}^0)$ to hitting a target set K_ε^n in minimum time τ_2^0 (defined by (4.2)). Moreover*

$$\sum_{i=1}^n \int_\Omega (y_i'(\tau^0;\mathbf{v}^0) - z_{id})\left(y_i'(\tau_2^0;\mathbf{v}) - y_i'(\tau_2^0;\mathbf{v}^0)\right)dx \geq 0 \quad \forall \mathbf{v}\in U_\epsilon^n. \tag{4.3}$$

Introduce the adjoint state $p(t;\mathbf{v}^0)$ by the solution of the following system

$$\left.\begin{array}{ll} \dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{v}^0) - \big(A(t)\mathbf{p}(t;\mathbf{v}^0)\big)_i = 0 & \text{in}\,\Omega\times]0,\tau_2^0[, \\ p_i(x,\tau_2^0;\mathbf{v}^0) = (y_i'(x,\tau_2^0;\mathbf{v}^0) - z_{id}) & \text{in}\,\Omega, \\ p_i'(x,\tau_2^0;\mathbf{v}^0) = 0 & \text{in}\,\Omega, \\ p_i(x,t;\mathbf{v}^0) = 0 & \text{in}\,\Gamma\times]0,\tau^0[. \end{array}\right\} \tag{4.4}$$

Since $(\,y_i'(x,\tau_2^0;\mathbf{v}^0) - z_{id}\,) \in L^2(\Omega)$, the existence of a unique solution for system (4.4) can be proved using transposition theorem (Theorem 2 with an obvious change of variables). We multiply the first equation in (4.4) by $y_i(t;\mathbf{v}) - y_i(t;\mathbf{v}^0)$ and integrate by parts from 0 to τ_2^0, we obtain the following identity:

$$\int_\Omega p_i(x,0;\mathbf{v}^0)(\mathbf{v}-\mathbf{v}^0)dx = \int_\Omega (y_i'(x,\tau_2^0;\mathbf{v}^0) - z_{id})(y_i'(x,\tau_2^0;\mathbf{v}) - y_i'(x,\tau_2^0;\mathbf{v}^0))dx.$$

The inequality (4.3) becomes

$$\sum_{i=1}^n \int_\Omega p_i(x,0;\mathbf{v}^0)(\mathbf{v}-\mathbf{v}^0)dx \geq 0 \quad \forall \mathbf{v} \in U_\epsilon^n. \tag{4.5}$$

We have thus proved;

Theorem 6. *We assume that (1.2) and (2.3) be held. Then there exist the adjoint state*

$$\mathbf{p} = (p_i)_{i=1}^n \in L^2(0,\tau_2^0;(H_0^1(\Omega))^n)$$

such that the optimal control $\mathbf{v^0}$ of problem (TOP2) is characterized by (4.4) and (4.5) together with (1.1) (with $v_i = v_i^0$, $i = 1,2,...,n$).

5. Position control-position observation problem

In this section, we consider the following third time-optimal control problem with control $\mathbf{u}$ acting in a position initial condition and a position observation $\mathbf{y}(x,t;\mathbf{u})$:

$$(TOP3): \qquad \min\{t:\ \mathbf{y}(x,t;\mathbf{u}) \in K_\varepsilon^n,\ \mathbf{u} \in U_\epsilon^n\}.$$

We assume that

$$\text{there exist a } \tau > 0 \text{ and } \ \mathbf{u} \in U_\epsilon^n \text{ with } \mathbf{y}(\tau;\mathbf{u}) \in K_\varepsilon^n \tag{5.1}$$

and if we set

$$\tau_3^0 = \inf\{\tau : \mathbf{y}(\tau;\mathbf{u}) \in K_\varepsilon^n \ \text{ fore some } \ \mathbf{u} \in U_\epsilon^n\}, \tag{5.2}$$

then in this case we can prove the following theorem;

Theorem 7. *If (1.2) and (2.3) are held, then there exist an admissible control $\mathbf{u}^0$ to the problem (TOP3), which steering $\mathbf{y}(t;\mathbf{u}^0)$ to hitting a target set K_ε^n in minimum time τ_3^0 (defined by (5.2)). Moreover*

$$\sum_{i=1}^n \int_\Omega (y_i(\tau_3^0;\mathbf{u}^0) - z_{id})\left(y_i(\tau_3^0;\mathbf{u}) - y_i(\tau_3^0;\mathbf{u}^0)\right)dx \geq 0 \quad \forall \mathbf{u} \in U_\epsilon^n, \tag{5.3}$$

which can be interpreted as the above sections to obtaining the following theorem:

Theorem 8. *We assume that* (1.2), (2.3) *as well as* (5.1) *and* (5.2) *are held. The time-optimal control* u^0 *of problem (TOP3) is characterized by the solution of the following systems of equations and inequalities:*

$$\left.\begin{aligned} \frac{\partial^2 y_i}{\partial t^2}(t;\mathbf{u}^0) - (A(t)y(t;\mathbf{u}^0))_i = f_i \quad & x \in \Omega, \quad t \in]0, \tau_3^0[, \\ y_i(x,0;\mathbf{u}^0) = u_i^0(x) \quad & x \in \Omega, \\ \frac{\partial y_i}{\partial t}(x,0;\mathbf{u}^0) = v_i(x) \quad & x \in \Omega, \\ y_i(x,t;\mathbf{u}^0) = 0, \quad & x \in \Gamma, \quad t \in]0, \tau_3^0[, \end{aligned}\right\} \tag{5.4}$$

$$\left.\begin{aligned} \frac{\partial^2 p_i}{\partial t^2}(t;\mathbf{u}^0) - (A(t)p(t;\mathbf{u}^0))_i = 0 \quad & x \in \Omega, \quad t \in]0, \tau_3^0[, \\ p_i(x,\tau_3^0;\mathbf{u}^0) = 0 \quad & x \in \Omega, \\ \frac{\partial p_i}{\partial t}(x,\tau_3^0;\mathbf{u}^0) = (y_i(x,\tau_3^0;\mathbf{u}^0) - z_{id}) \quad & x \in \Omega, \\ p_i(x,t;\mathbf{u}^0) = 0, \quad & x \in \Gamma, \quad t \in]0, \tau_3^0[, \end{aligned}\right\} \tag{5.5}$$

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial p_i}{\partial t}(x,0;\mathbf{u}^0)(v_i - v_i^0)dx \geq 0 \quad \forall \mathbf{v} = (v_1, v_2, ..., v_n) \in U_\epsilon^n.$$

together with

$$\left.\begin{aligned} & p_i(t;u^0) \in L^2\left(0,\tau_3^0; H_0^1(\Omega)\right), \\ & y_i(t;u^0), \ \frac{\partial p_i}{\partial t}(t;u^0) \in L^2\left(0,\tau_3^0; L^2(\Omega)\right). \end{aligned}\right\} \tag{5.6}$$

6. Position control - velocity observation problem

In this section, we consider the following fourth time-optimal control problem with a control $\mathbf{u}$ acting in a position initial condition and a velocity observation $\mathbf{y}(x,t;\mathbf{u})$:

$$(TOP4): \qquad \min\{t: \ \mathbf{y}'(x,t;\mathbf{u}) \in K_\varepsilon^n, \ \mathbf{u} \in U_\epsilon^n\}.$$

We assume that

$$\text{there exist a } \tau > 0 \text{ and } \ \mathbf{u} \in U_\epsilon^n \text{ with } \mathbf{y}'(\tau;\mathbf{u}) \in K_\varepsilon^n \tag{6.1}$$

and if we set

$$\tau_4^0 = \inf\{\tau : \mathbf{y}'(\tau;\mathbf{u}) \in K_\varepsilon^n \ \text{ fore some } \ \mathbf{u} \in U_\epsilon^n\}, \tag{6.2}$$

then we can prove the following theorem:

Theorem 9. *If* (1.2), (2.3) *and* (6.1) *are held, then there exists an admissible control* $\mathbf{u}^0$ *to the problem (TOP4), which steering* $\mathbf{y}'(t;\mathbf{u}^0)$ *to hitting a target set* K_ε^n *in a minimum time* τ_4^0 *(defined by* (6.2)*). Moreover*

$$\sum_{i=1}^{n} \int_\Omega (y_i'(\tau_4^0;\mathbf{u}^0) - z_{id})\left(y_i'(\tau_4^0;\mathbf{u}) - y_i'(\tau_4^0;\mathbf{u}^0)\right) dx \geq 0 \quad \forall \mathbf{u} \in U_\epsilon^n, \tag{6.3}$$

which can be interpreted as the above sections to obtaining the following theorem:

Theorem 10. *We assume that (1.2) and (2.3) hold. The time-optimal control* $\mathbf{u}^0$ *of problem (TOP4) is characterized by the solution of the following systems of equations and inequalities:*

$$\left.\begin{array}{lll} \dfrac{\partial^2 y_i}{\partial t^2}(t;\mathbf{u}^0) - (A(t)y(t;\mathbf{u}^0))_i = f_i & x \in \Omega, & t \in]0,\tau_4^0[, \\ y_i(x,0;\mathbf{u}^0) = u_i^0(x) & x \in \Omega, & \\ \dfrac{\partial y_i}{\partial t}(x,0;\mathbf{u}^0) = v_i(x) & x \in \Omega, & \\ y_i(x,t;\mathbf{u}^0) = 0, & x \in \Gamma, & t \in]0,\tau_4^0[, \end{array}\right\} \tag{6.4}$$

$$\left.\begin{array}{lll} \dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{u}^0) - (A(t)p(t;\mathbf{u}^0))_i = 0 & x \in \Omega, & t \in]0,\tau_4^0[, \\ p_i(x,\tau_4^0;\mathbf{u}^0) = (y_i'(x,\tau_4^0;\mathbf{u}^0) - z_{id}) & x \in \Omega, & \\ \dfrac{\partial p_i}{\partial t}(x,\tau_4^0;\mathbf{u}^0) = 0 & x \in \Omega, & \\ p_i(x,t;\mathbf{u}^0) = 0, & x \in \Gamma, & t \in]0,\tau_4^0[, \end{array}\right\} \tag{6.5}$$

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial p_i}{\partial t}(x,0;\mathbf{u}^0)(u_i - u_i^0)dx \geq 0 \quad \forall \mathbf{u} = (u_1,u_2,...,u_n) \in U_\epsilon^n,$$

together with

$$y_i(t;\mathbf{u}^0),\ p_i(t;\mathbf{u}^0) \in L^2\left(0,\tau_4^0;L^2(\Omega)\right). \tag{6.6}$$

7. Comments

- We note that, in this paper, we have chosen to treat a special systems involving Laplace operator, just for the simplicity. Most of the results we described in this paper apply, without any change on the results, to more general hyperbolic systems involving the following second order operator :

$$L(x,\cdot) = \sum_{i,j=1}^{n} b_{ij}(x,\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,\cdot)\frac{\partial}{\partial x_j} + b_0(x,\cdot)$$

with sufficiently smooth coefficients (in particular, $b_{ij},\, b_j,\, b_0 \in L^\infty(Q),\ \ b_j,\, b_0 > 0$) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^{n} \eta_i\eta_j \geq \sigma \sum_{i=1}^{n} \eta_i \quad \forall (x,t) \in Q$$

for all $\eta_i \in \mathbb{R}$ and some constant $\sigma > 0$.

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator L (see [9]).

- In this paper, we have chosen to treat a co-operative hyperbolic systems with Dirichlet boundary conditions. The results can be extended to the case of $n \times n$ co-operative hyperbolic system with Neumann boundary conditions: If we take $H^1(\Omega)$ instead of $H_0^1(\Omega)$, we have to replace the Dirichlet boundary conditions $y_i = 0,\, p_i = 0$ on the boundary by Neumann boundary conditions $\frac{\partial y_i}{\partial \nu} = 0,\ \frac{\partial p_i}{\partial \nu} = 0$, where ν is the outward normal.

• In this paper, we take a simple target set K^n_ε . In (TOP1) (for example), if we take

$$K^n_\varepsilon = \Big\{ z \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} + \sum_{j=1}^{N} \|\frac{\partial z_i}{\partial x_j} - z_{id}\|_{L^2(\Omega)} \le \epsilon \Big\},$$

then the necessary optimality conditions coincide with (3.10), (3.11), (1.1) (with $v_i = v_i^0$, $i = 1, 2$) and $(y_i(x, \tau_1^0; \mathbf{v}^0) - z_{id})$ in (3.10) is replaced by $(-\Delta_x + I)(y_i(x, \tau_1^0; \mathbf{v}^0) - z_{id})$

• The results in this paper, carry over to the optimal control problems with fixed-time ([12] Chapter 4), for example, the results of (TOP1) carry over to the fixed-time problem

$$\text{minimize} \sum_{i=1}^{n} \int_\Omega |y_i(x, T; u) - z_{id}(x)|^2 dx, \quad T \text{ fixed },$$

subject to (1.1) [except in the trivial case where $z_{id}(x) = y_i(x, T; \mathbf{v})$ for some admissible control $\mathbf{v} = (v_i)_{i=1}^n$.] This can proven in an analogous manner, as the necessary and sufficient conditions for the optimality of this problem coincide with (3.10), (3.11) and (1.1) (with $v_i = v_i^0$, $i = 1, 2, ..., n$).

• As a final comment, we note that the control problem for the second order evolution system (1.1) can be reduced to a similar control problem first order system; in the usual way : set $\psi = \begin{pmatrix} \mathbf{y} \\ \frac{\partial \mathbf{y}}{\partial t} \end{pmatrix}$ and rewrite (1.1) in the first order form. However, the existing results on the time-optimal problem ([7], [8], [12]) pertain to the case, where the observation is only one case (the position-velocity), but here we can take different cases.

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Bangmod International
Journal of Mathematical Computational Science
ISSN: 2408-154X
Bangmod-JMCS Online @ http://bangmod-jmcs.kmutt.ac.th/

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