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BALL COMPARISON FOR VARIANTS OF CHEBYSHEV'S METHOD WITH THIRD OR FOURTH ORDER OF CONVERGENCE

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Abstract We present a local convergence analysis for some multipoint variants of Chebyshev's method. These methods use three function evaluations at each step. The convergence order of these methods is shown to be three or four. In earlier papers the local convergence of these methods is shown under hypotheses up to the fifth derivative of the function although only the first derivative appears in these methods. In the present paper we show the convergence using only the first derivative. The applicability of these methods is expanded in this way. Computable radii of convergence and error bounds are provided using Lipschitz constants. Numerical examples where our result apply but earlier ones do not apply to solve equations are also given in this study.

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1. Introduction

In this paper the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \tag{1.1}$$

is analyzed. Here $F : D \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear function, D is a convex subset of $\mathbb{R}$. Newton-like methods are widely used for finding solution of (1.1). These methods are

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usually studied based on: semi-local and local convergence. The semi-local convergence method is based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [3, 4, 22–24, 26].

Third order methods such as Euler's, Halley's, super-Halley's, Chebyshev's [1]-[31] require the evaluation of the second derivative F'' at each step, which in general is very expensive. That is why many authors have used higher order multi-point methods [1]-[31]. In this paper, we present the local convergence of two multipoint variants of Chebyshev's method defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - \beta F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - F'(x_n)^{-1}F(x_n) \\ &\quad \times[1 + \beta^2 F(x_n)\frac{F(y_n) - (1-\beta)F(x_n)}{(\gamma F(x_n) - 2\alpha F(y_n))^2}] \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= x_n - \beta F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - F'(x_n)^{-1}F(x_n)[1 + \frac{F'(x_n)(F'(x_n) - F'(y_n))}{2((\beta-\alpha)F'(x_n) + \alpha F'(y_n))^2}] \end{aligned} \tag{1.3}$$

where x_0 is an initial point, $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma = \beta^2 + 2\alpha(1-\beta)$. These methods were derived from Chebyshev's method in [5]. The local convergence of these methods was shown under hypotheses reaching up to the fifth derivative of function F although only the first derivative appears in these methods. In particular, they showed that for method (1.2):

- the convergence order is three, for $\alpha \neq \frac{1}{2}$ and $\beta = 1$, $\alpha = \frac{1}{2}$ and $\beta \neq 1$, $\alpha \neq \frac{1}{2}$ and $\beta \neq 1$.
- The convergence order is four, if $\alpha = \frac{1}{2}$ and $\beta = 1$.

method (1.3):

- The convergence order is three, $\alpha \neq \frac{1}{2}$ and $\beta = \frac{2}{3}$, $\alpha = \frac{1}{2}$ and $\beta \neq \frac{2}{3}$, $\alpha \neq \frac{1}{2}$ and $\beta = \frac{2}{3}$.
- The convergence order is four, for $\alpha = \frac{1}{2}$ and $\beta = \frac{2}{3}$.

However, the hypotheses on the $F^i(x), i = 1, 2, 3, 4, 5$ limit the applicability of these methods. As a motivational let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, function f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative. This way we expand the applicability of these methods. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathbb{R}$ or $\mathbb{C}$ [1]–[31]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*.

But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) and method (1.3) in Section 2. The same technique can be used to other methods. Our results are also presented in affine invariant form.

The rest of the paper is organized as follows. The local convergence of methods (1.2) and method (1.3) are given in Section 2, whereas the numerical examples are given in the concluding Section 3.

2. Local Convergence Analysis

We present the local convergence analysis of method (1.2) and method (1.3) in this section. Let $L_0 > 0$, $L > 0$, $M \geq 1, \alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R} - \{0\}$ be given parameters. It is convenient for the local convergence analysis of method (1.2) that follows to define some functions and parameters. Define functions g_1, p, h_p on the interval $[0, +\infty)$ by:

$$g_1(t) = \frac{1}{2(1-L_0t)}(Lt + 2M|1-\beta|), \tag{2.1}$$

$$p(t) = \frac{1}{|\gamma|}[\frac{|\gamma|L_0}{2}t + 2|\alpha|Mg_1(t)],$$

$$h_p(t) = p(t) - 1$$

and parameters r_1 and r_A by

$$r_1 = \frac{2(1-M|1-\beta|)}{2L_0 + L}, \quad r_A = \frac{2}{2L_0 + L}.$$

Suppose that

$$M|1-\beta| < 1 \tag{2.2}$$

Then, we have that $0 < r_1 < r_A$ and for each $t \in [0, r_1)$, $0 \leq g_1(t) < 1$. Suppose that

$$2|\alpha|M^2|1-\beta| < |\gamma|. \tag{2.3}$$

We get by (2.3) that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows from the intermediate value theorem that function h_p has zeros in the interval $[0, \frac{1}{L_0})$. Denote by r_p the smallest such zero. Define functions g_2 and h_2 on the interval $[0, r_p)$ by

$$g_2(t) = \frac{1}{2(1-L_0t)}[L + \frac{2\beta^2M^3(g_1(t) + |1-\beta|)}{|\gamma|^2(1-p(t))^2}]t \tag{2.4}$$

and

$$h_2(t) = g_2(t) - 1.$$

Then, we have that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2 the smallest such zero.

Define

$$r = \min\{r_1, r_2\}. \tag{2.5}$$

Then, we have $0 < r < r_A$ and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1 \tag{2.6}$$

$$0 \leq p(t) < 1 \tag{2.7}$$

and

$$0 \le g_2(t) < 1. \tag{2.8}$$

Denote by $U(v,\rho)$, $\bar{U}(v,\rho)$, respectively for the open and closed balls in $\mathbb{R}$ with center $v \in \mathbb{R}$ and of radius $\rho > 0$. Next, we present the local convergence of method (1.2) using the preceding notation.

THEOREM 2.1. Let $F : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M \ge 1, \alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{R} - \{0\}$ such that for each $x, y \in D$, (2.2), (2.4),

$$F(x^*) = 0,\ F'(x^*) \ne 0, \tag{2.9}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \le L_0|x - x^*|, \tag{2.10}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \le L|x - y|, \tag{2.11}$$

$$|F'(x^*)^{-1}F'(x)| \le M, \tag{2.12}$$

and

$$\bar{U}(x^*, \bar{r}) \subseteq D, \tag{2.13}$$

hold, where the radius r is given by (2.5). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, ...$ and converges to x^*. Moreover the following estimates hold

$$|y_n - x^*| \le g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r \tag{2.14}$$

and

$$|x_{n+1} - x^*| \le g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.15}$$

where functions g_1 and g_2 are defined by (2.1) and (2.4), respectively . Furthermore, for $T \in [r, 2/L_0)$ the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof. We shall show estimates (2.14) and (2.15) using mathematical induction. By hypothesis, $x_0 \in U(x^*, r) - \{x^*\}$ and (2.10), we get that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \le L_0|x_0 - x^*| < L_0 r < 1. \tag{2.16}$$

It follows from (2.16) and the Banach Lemma on invertible functions [3, 4, 25, 28], that $F'(x_0) \ne 0$ and

$$|F'(x_0)^{-1}F'(x^*)| \le \frac{1}{1 - L_0|x_0 - x^*|}. \tag{2.17}$$

Hence y_0 is well defined by the first sub-step of method (1.2) for $n = 0$. We can write by (2.9) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.18}$$

Notice that $|x^*+\theta(x_0-x^*)-x^*| = \theta|x_0-x^*| \le |x_0-x^*| < 1$. That is $x^*+\theta(x_0-x^*) \in U(x^*, r)$. Using (2.12) and (2.18), we have that

$$|F'(x^*)^{-1}F(x_0)| = |\int_0^1 F'(x^*)^{-1}F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta| \le M|x_0-x^*|. \tag{2.19}$$

In view (2.5), (2.6), (2.9), (2.17), (2.19) and the first sub-step of method (1.2) for $n = 0$, we obtain in turn that

$$\begin{aligned} |y_0-x^*| &\le |x_0-x^*-F'(x_0)^{-1}F(x_0| \\ &\quad +|1-\beta||F'(x_0)^{-1}F(x_0)| \\ &\le |F'(x_0)^{-1}F'(x^*)||\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*))-F'(x_0))(x_0-x^*)d\theta| \\ &\quad +|1-\beta||F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)| \\ &\le \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)} + \frac{|1-\beta|M|x_0-x^*|}{1-L_0|x_0-x^*|} \\ &= g_1(|x_0-x^*|)|x_0-x^*| < |x_0-x^*| < r, \end{aligned} \tag{2.20}$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*, r)$. Then, as in (2.19), we have that

$$|F'(x^*)^{-1}F(y_0)| \le M|y_0-x^*| \le Mg_1(|x_0-x^*|)|x_0-x^*|. \tag{2.21}$$

Next, we must show that $\gamma F(x_0) - 2\alpha F(y_0) \ne 0$. Using (2.5)), (2.7), (2.10), (2.21) and the triangle inequality, we have in turn that

$$\begin{aligned} & |(\gamma F'(x^*)(x_0-x^*))^{-1}[|\gamma(F(x_0)-F(x^*)-F'(x^*)(x_0-x^*))-2\alpha F(y_0)| \\ \le\ & (|\gamma||x_0-x^*|)^{-1}[|\gamma||F(x^*)^{-1}(F(x_0)-F(x^*)-F'(x^*)(x_0-x^*)| \\ & 2|\alpha||F'(x^*)^{-1}F(y_0)|] \\ \le\ & (|\gamma||x_0-x^*|)^{-1}[\frac{|\gamma|L_0|x_0-x^*|^2}{2} + 2|\alpha|Mg_1(|x_0-x^*|)|x_0-x^*|] \\ =\ & p(|x_0-x^*|) < p(r) < 1. \end{aligned} \tag{2.22}$$

Hence, we get that

$$|(\gamma F(x_0)-2\alpha F(y_0))^{-1}F'(x^*)| \le \frac{1}{|\gamma||x_0-x^*|(1-p(|x_0-x^*|))}. \tag{2.23}$$

That is x_1 is well defined by the second sub-step of method (1.2) for $n = 0$. Then, using (2.5), (2.7), (2.17), (2.20), (2.21) and (2.23), we get in turn that

$$\begin{aligned} |x_1-x^*| &\le |x_0-x^*-F'(x_0)^{-1}F(x_0)| \\ &\quad +|\frac{\beta^2F'(x_0)^{-1}F'(x^*)(F'(x^*)^{-1}F(x_0))^2[F'(x^*)^{-1}F(y_0)-(1-\beta)F'(x^*)^{-1}F(x_0)]}{(F'(x^*)^{-1}(\gamma F(x_0)-2\alpha F(y_0)))^2}| \\ &\le \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)} + \frac{2\beta^2M^3|x_0-x^*|^2[|y_0-x^*|+M|1-\beta||x_0-x^*|]}{2(1-L_0|x_0-x^*|)|\gamma|^2|x_0-x^*|(1-p(|x_0-x^*|))^2} \\ &\le g_2(|x_0-x^*|)|x_0-x^*| < |x_0-x^*| < r, \end{aligned}$$

which shows (2.15) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at estimates (2.14) and (2.15). Using the estimate $|x_{k+1}-x^*| < |x_k-x^*| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$.

To show the uniqueness part, let $B = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.10) we get that

$$\begin{aligned} |F'(x^*)^{-1}(B - F'(x^*))| &\leq \int_0^1 L_0 |y^* + \theta(x^* - y^*) - x^*| d\theta \\ &\leq L_0 \int_0^1 (1-\theta)|x^* - y^*| d\theta \leq \frac{L_0}{2} T < 1. \end{aligned} \tag{2.24}$$

It follows from (2.24) and the Banach Lemma on invertible functions that B is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = B(x^* - y^*)$, we deduce that $x^* = y^*$.

□

In order for us to study the local convergence of method (1.3), we need to define more scalar functions and parameters. Define functions q and h_q on the interval $[0, \frac{1}{L_0})$ for $\beta \neq 0$ by

$$q(t) = \frac{L_0}{|\beta|}(|\beta - \alpha| + |\alpha| g_1(t))t$$

and

$$h_q(t) = q(t) - 1.$$

We have that $h_q(0) = -1 < 0$ and $h_q(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Denote by r_q the smallest zero of function h_q in the interval $(0, \frac{1}{L_0})$. Moreover, define functions g_3 and h_3 on the interval $[0, r_q)$ by

$$g_3(t) = \frac{1}{2(1 - L_0 t)}[L + \frac{L_0 M(1 + L_0 t)(1 + g_1(t))}{|\beta|^2(1 - q(t))^2}]t \tag{2.25}$$

and

$$h_3(t) = g_3(t) - 1.$$

We get that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_q^-$. Denote by r_3 the smallest zero of function h_3 in the interval $(0, r_q)$. Set

$$\bar{r} = \min\{r_1, r_3\}. \tag{2.26}$$

Using the preceding notation and the proof of Theorem 2.1, we arrive at the following local result for method (1.3).

THEOREM 2.2. Let $F : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M \geq 1, \alpha \in \mathbb{R}, \beta \neq 0$ such that hypotheses (2.2), (2.9)–2.12) and

$$\bar{U}(x^*, \bar{r}) \subseteq D \tag{2.27}$$

hold, where the radius $\bar{r}$ is given by (2.26). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.3) is well defined, remains in $U(x^*, \bar{r})$ for each $n = 0, 1, 2, ...$ and converges to x^*. Moreover, the estimate (2.14) and

$$|x_{n+1} - x^*| \leq g_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{2.28}$$

hold, where the function g_3 is defined by (2.25). Furthermore, for $T \in [\bar{r}, 2/L_0)$ the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof. According to Theorem 2.1 we only need to show estimate (2.28). We must show that $(\beta-\alpha)F'(x_0)+\alpha F'(y_0)\neq 0$. Using (2.10), (2.20) and (2.26), we get in turn that

$$\begin{aligned}
& |(\beta F'(x^*))^{-1}[(\beta-\alpha)(F'(x_0)-F'(x^*))+\alpha(F'(y_0)-F'(x^*))]| \\
\leq\; & |\beta|^{-1}[|\beta-\alpha|L_0|x_0-x^*|+|\alpha|L_0|y_0-x^*|] \\
\leq\; & q(|x_0-x^*|)<q(r)<1. \qquad (2.29)
\end{aligned}$$

It follows from (2.29) that

$$|((\beta-\alpha)F'(x_0)+\alpha F'(y_0))^{-1}F'(x^*)|\leq \frac{1}{|\beta|(1-q(|x_0-x^*|))}. \qquad (2.30)$$

Hence, x_1 is well defined by the second sub-step of method (1.3) for $n=0$. Then, using (2.10), (2.20), (2.26) and (2.30) we get in turn that

$$\begin{aligned}
& |x_1-x^*| \\
\leq\; & |x_0-x^*-F'(x_0)^{-1}F(x_0)| \\
& +\frac{1}{2}|(F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F(x_0))| \\
& \times\frac{[F'(x^*)^{-1}(F'(x_0)-F'(x^*))+I][F'(x^*)^{-1}(F(x_0)-F'(x^*))+F'(x^*)^{-1}(F'(x^*)-F'(y_0))]}{[F'(x^*)^{-1}((\beta-\alpha)F'(x_0)+\alpha F'(y_0))]^2}| \\
\leq\; & \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)}+\frac{L_0M(1+L_0|x_0-x^*|)(1+g_1(|x_0-x^*|))|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)|\beta|^2(1-q(|x_0-x^*|))} \\
\leq\; & g_3(|x_0-x^*|)|x_0-x^*|<|x_0-x^*|<\bar{r},
\end{aligned}$$

which shows (2.28) for $n=0$ and $x_1\in U(x^*,\bar{r})$.

□

REMARK 2.3. The iterates y_n of method (1.3) (similarly for method (1.2)) do not have to belong in $U(x^*,\bar{r})$ for the sequence $\{x_n\}$ to converge to x^*. Condition (2.2) can be dropped as follows. Define

$$\bar{r}_1=g_1(r_3)r_3$$

and set

$$r^*=\min\{\bar{r}_1,r_3\}.$$

Then, we arrive at:

THEOREM 2.4. Suppose that the hypotheses of Theorem 2.2 hold except (2.2) and r^* replacing $\bar{r}$. Then, the conclusions hold but (2.14) is replaced by

$$|y_n-x^*|\leq g_1(|x_n-x^*|)|x_n-x^*|\leq g_1(r_3)r_3=\bar{r}_1. \qquad (2.31)$$

REMARK 2.5. 1. In view of (2.10) and the estimate

$$\begin{aligned}
|F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x)-F'(x^*))+I| \\
&\leq 1+|F'(x^*)^{-1}(F'(x)-F'(x^*))|\leq 1+L_0|x-x^*|
\end{aligned}$$

condition (2.12) can be dropped and M can be replaced by

$$M(t)=1+L_0t,\ \text{ for each } t\in[0,\frac{1}{L_0})$$

or simply by $M=2$, since $t\in[0,\frac{1}{L_0})$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. The radius r_A was shown by us to be the convergence radius of Newton's method [3]-[4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.32}$$

under the conditions (2.10) and (2.11). The convergence radius r of the method (1.2) or r^* of method (1.3) cannot be larger than the convergence radius r_A of the second order Newton's method. As already noted in [3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [22]

$$r_R = \frac{2}{3L}. \tag{2.33}$$

In particular, for $L_0 < L$ we have that

$$r_R < r$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \;\; as \;\; \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [22].

4. It is worth noticing that method (1.2) or method (1.3) are not changing when we use the conditions of Theorem 2.1 or the conditions of Theorem 2.2 instead of the stronger conditions used in [5]–[31]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

3. Numerical Examples

We present numerical examples in this section.

EXAMPLE 3.1. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = N = 146.6629073$, $M = 2$. The parameters for methods (1.2) and (1.3) are

$$\alpha = 0.1,\; \beta = 1.5,\; \gamma = -4,\; r_A = 0.0045,\; r_1 = 0.0091,$$

$$r_p = 0.0061,\; r_2 = 0.0048,\; r_q = 0.0070,\; r_3 = 0.1279,\; \bar{r}_1 = 0.1279.$$

EXAMPLE 3.2. Let $D = (-\infty, +\infty)$. Define function f of D by

$$f(x) = \sin(x). \tag{3.1}$$

Then we have for $x^* = 0$ that $L_0 = L = M = 1$. The parameters for methods (1.2) and (1.3) are

$$\alpha = 1,\ \beta = 0.5,\ \gamma = 1.75,\ r_A = 0.6667,\ r_1 = 0.3333,$$

$$r_p = 0.2184,\ r_2 = 0.3049,\ r_q = 0.3333,\ r_3 = 0.0820,\ \bar{r}_1 = 0.0483.$$

EXAMPLE 3.3. Let $D = [-1, 1]$. Define function f of D by

$$f(x) = e^x - 1. \tag{3.2}$$

Using (3.2) and $x^* = 0$, we get that $L_0 = e - 1 < L = M = e$.The parameters for methods (1.2) and (1.3) are

$$\alpha = 0.1,\ \beta = 1.75,\ \gamma = 7.5,\ r_A = 0.3249,\ r_1 = 0.8124,$$

$$r_p = 0.6256,\ r_2 = 0.0977,\ r_q = 0.5983,\ r_3 = 0.4791,\ \bar{r}_1 = 5.8303.$$

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