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BALL CONVERGENCE FOR A COMPUTATIONALLY EFFICIENT FIFTH-ORDER METHOD FOR SOLVING EQUATIONS IN BANACH SPACE UNDER WEAK CONDITIONS

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Abstract In the present paper, we consider a fifth order method considered in Jaiswal (2016) to solve equations in Banach space under weaker assumptions. Using the idea of restricted convergence domains we extend the applicability of the method considered by Jaiswal (2016). Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

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1. Introduction

Let X and Y be Banach spaces and let $U(a,\rho)$, $\overline{U}(a,\rho)$ stand respectively, for the open and closed balls in X, with center $a \in X$ and of radius $\rho > 0$. Consider the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

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where $F : D \subseteq X \longrightarrow Y$ is a Fréchet-differentiable operator defined on a convex subset D of X. Higher order multi-point methods are studied in the literature (see [6–16]) for approximating the solution x^* of (1.1). In order to reduce the computational cost of these higher order method many researchers try to obtained a general law to accelerate the convergence for all the iterative methods which use Newton iteration as a predictor. For example, Kou et.al [12] and Cordero et al. [5–7] introduced a the following construction,

$$\begin{aligned} z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k), \end{aligned}$$

where $y_k = x_k - F'(x_k)^{-1}F(x_k)$ and ϕ is the iteration function. For using extended Newton iteration as a predictor and accelerating the order of convergence the following construction is introduced in [17]

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k)' \\ z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - \{2[\frac{1}{2a}F'(y_k) + (1-\frac{1}{2a})F'(x_k)]^{-1} - F'(x_k)^{-1}]F(z_k). \end{aligned}$$

In the present paper, we consider the following construction considered in [8]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - 5F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - \frac{1}{5}F'(x_n)^{-1}(-16F(y_n) + F(z_n)), \end{aligned} \tag{1.2}$$

where x_0 is an initial point. Our goal is to weaken the assumptions in [8], so that the applicability of the method (1.2) can be improved.

In earlier studies such as [5–17], higher order methods are considered for approximating the solution x^* of (1.1). But, for the convergence analysis of these methods, in addition to the assumptions on F' and F''; require the assumptions of the form

$$\|F'''(x) - F'''(y)\| \le L\|x-y\|, \;\; x, y \in \Omega, \;\; L \ge 0 \tag{1.3}$$

or

$$\|F'''(x) - F'''(y)\| \le w(\|x-y\|), \;\; x, y \in \Omega, \tag{1.4}$$

where $w(z)$ is a nondecreasing continuous function for $z > 0$ and $w(0) = 0$ (see [17]).

A typical example of (1.1) that does not satisfies (1.3) or (1.4) is the following equation

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{1.5}$$

where $F : [-\frac{5}{2}, \frac{1}{2}] \longrightarrow \mathbb{R}$. We have that

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2$$

$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Then, obviously, function F''' is unbounded on Ω.

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result. Special cases and numerical examples are given in the last section.

2. Ball convergence

The ball convergence analysis of method (1.2) is based on some functions and parameters. Let w_0 be a continuous, non-negative, non-decreasing function defined on the interval $[0,+\infty)$ with $w_0(0) = 0$. Define the parameter r_0 by

$$r_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Let also w, v continuous, nonnegative, nondecreasing function defined on the interval $[0,r_0)$ with $w(0) = 0$. Define functions $f_i, g_i,\ i = 1, 2, 3$ on the interval $[0, r_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)},$$

$$g_2(t) = \left(1 + \frac{5\int_0^1 v(\theta g_1(t)t)d\theta}{1-w_0(t)}\right) g_1(t),$$

$$g_3(t) = g_2(t) + \frac{1}{5(1-w_0(t))}\left[16\int_0^1 v(\theta g_1(t)t)g_1(t)d\theta + \int_0^1 v(\theta g_2(t)t)d\theta g_2(t)\right]$$

and

$$h_1(t) = g_1(t) - 1.$$

We have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_0^-$. It then follows from the intermediate value theorem that function h_1 has zeros in the interval $(0, r_0)$. Denote by r_1 the smallest such zero. Then, we get $h_2(0) = -1 < 0$ and $h_2(r_1) = \frac{5\int_0^1 v(\theta r_1)d\theta}{1-w_0(r_1)} > 0$, since $g_1(r_1) = 1$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, r_1)$. We also have that $h_3(0) = -1 < 0$ and $h_3(t) \longrightarrow +\infty$ as $t \to r_0^-$. Denote by r_3 the smallest zero of functions h_3 on the interval $(0,r_0)$.

Define the radius of convergence r by

$$r = \min\{r_2, r_3\}. \tag{2.2}$$

Then, we have that for each $t \in [0, r)$

$$0 \leq g_i(t) < 1. \tag{2.3}$$

Next, we present the ball convergence analysis of method (1.2) using the preceding notation.

THEOREM 2.1. Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator. Let w_0 be a continuous, non-negative, non-decreasing function defined on the interval $[0, +\infty)$ with $w_0(0) = 0$. Suppose:

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X); \tag{2.4}$$

and for each $x \in D$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \leq w_0(\|x - x^*\|); \tag{2.5}$$

there exist continuous, non-negative, non-decreasing functions w, v defined on the interval $[0, r_0)$ such that for each $x, y \in D_0 := D \cup U(x^*, r_0)$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y)\| \leq w(\|x - y\|), \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x-y\|), \tag{2.7}$$

and

$$U(x^*,r) \subseteq D, \tag{2.8}$$

where the r_0 and r are defined by (2.1) and (2.2), respectively. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*,r)-\{x^*\}$ by method (1.2) is well defined in $U(x^*,r)$, remains in $U(x^*,r)$ for each $n=0,1,2,\ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\| < r, \tag{2.9}$$

$$\|z_n - x^*\| \leq g_2(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\|, \tag{2.10}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\|, \tag{2.11}$$

where, the functions $g_i, i=1,2,3$ are defined previously. Furthermore, if there exists $R \geq r_0$ such that

$$\int_0^1 w_0(\theta R)d\theta < 1, \tag{2.12}$$

then the limit point x^* is the only solution of equation $F(x)=0$ in $D_1 = D \cap U(x^*,R)$.

Proof. Using mathematical induction, we shall show that the sequence $\{x_n\}$ generated by (1.2) is well defined in $U(x^*,r)$, remains in $U(x^*,r)$ for each $n=0,1,2,\cdots$ and converges to x^* so that estimates (2.9)-(2.11) are satisfied. By hypotheses $x_0 \in U(x^*,r)-\{x^*\}$, (2.1) and (2.5), we have that

$$\|F'(x^*)^{-1}(F'(x_0)-F'(x^*)\| \leq w_0(\|x_0-x^*\|) \leq w_0(r) < 1. \tag{2.13}$$

It follows from (2.13) and the Banach Lemma on invertible operators [1, 14, 16] that $F'(x)^{-1} \in L(Y,X)$, y_0 is well defined by the first substep of method (1.2) and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1-w_0(\|x_0-x^*\|)}. \tag{2.14}$$

We can write the identity,

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{2.15}$$

Using (2.1), (2.2), (2.3) (for $i=1$), (2.6), (2.14) and (2.15), we get in turn that

$$\begin{aligned} \|y_0-x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|\textstyle\int_0^1 F'(x^*)^{-1}[F'(x^*+\theta(x_0-x^*)) \\ &\quad -F'(x_0)](x_0-x^*)d\theta\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)}, \\ &= g_1(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \end{aligned} \tag{2.16}$$

which shows (2.9) for $n=0$ and $y_0 \in U(x^*,r)$.

We can write by (2.4) that

$$F(y_0) = F(y_0) - F(x^*) = \int_0^1 F'(x^* + \theta(y_0 - x^*))(y_0 - x^*)d\theta. \tag{2.17}$$

Notice that $x^* + \theta(y_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0,1]$. Using (2.7) and (2.17), we get that

$$\begin{aligned} \|F'(x^*)^{-1}F(y_0)\| &\leq \int_0^1 v(\theta\|y_0 - x^*\|)d\theta\|y_0 - x^*\| \\ &\leq \int_0^1 v(\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \end{aligned} \tag{2.18}$$

Using (2.2), (2.3) (for $i = 2$,), (2.6), (2.14), (2.16), (2.17) and the second substep of method (1.2) for $n = 0$, we obtain that

$$\begin{aligned} \|z_0 - x^*\| &\leq \|y_0 - x^*\| + 5\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\| \\ &\leq \|y_0 - x^*\| + 5\frac{\int_0^1 v(\theta\|y_0 - x^*\|)\|y_0 - x^*\|d\theta}{1 - w_0(\|x_0 - x^*\|)} \\ &\leq (1 + +5\frac{\int_0^1 v(\theta\|y_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)})\|y_0 - x^*\| \\ &\leq g_2(\|y_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.19}$$

which shows (2.10) for $n = 0$ and $z_0 \in U(x^*, r)$. In view of (2.2), (2.3) (for $i = 3$), (2.14), (2.17) (for $z_0 = x_0$) and (2.18), we have in turn that

$$\begin{aligned} \|x_1 - x^*\| &= \|z_0 - x^*\| + \tfrac{1}{5}\|F'(x_0)^{-1}F'(x^*)\| \\ &\quad [16\|F'(x^*)^{-1}F(y_0)\| + \|F'(x^*)^{-1}F'(z_0)\|] \\ &\leq \|z_0 - x^*\| + \frac{1}{5(1 - w_0(\|x_0 - x^*\|)}[16\|\int_0^1 v(\theta g_1(\|x_0 - x^*\|)d\theta)\|x_0 - x^*\|g_1(\|x_0 - x^* \\ &\quad + \int_0^1 v(\theta g_2(\|x_0 - x^*\|\|x_0 - x^*\|)d\theta g_2(\|x_0 - x^*\|)]\|x_0 - x^*\| \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.20}$$

which shows (2.11) for $n = 0$ and $x_1 \in U(x^*, r)$. Simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates, we arrive at estimates (2.9)-(2.11). Then, from the estimates

$$\|x_{n+1} - x^*\| \leq c\|x_k - x^*\| < r, \tag{2.21}$$

where $c = g_3(\|x_0 - x^*\|) \in [0,1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally to show the uniqueness part, let $T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$ where $y^* \in D_2$

with $F(y^*) = 0$. Using (2.13), we obtain that

$$\begin{aligned} \|F'(x^*)^{-1}(T - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \\ &\leq w_0(\theta R)d\theta < 1, \end{aligned} \tag{2.22}$$

Hence, we have that $T^{-1} \in L(Y, X)$. Then, from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$, we conclude that $x^* = y^*$. □

REMARK 2.2. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [1] as the convergence radius for Newton's method under condition (2.7)-(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [14] and Traub [16] is given by

$$\rho = \frac{2}{3L} < r_1.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = U(0, 1)$. Then, we have that $L_0 = e - 1 < l = e$, so $\rho = 0.24252961 < r_1 = 0.324947231$.

Moreover, the new error bounds [1] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [4, 6]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r to be larger than r_1 (see (2.3)) .

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [1–4].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [1–4]:

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [3–8, 15, 17]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each} \quad n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (2.5) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|)\end{aligned}$$

condition (2.7) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t)$$

or

$$v(t) = 1 + w_0(r_0),$$

since $t \in [0, r_0)$.

3. Numerical Examples

The numerical examples are presented in this section.

EXAMPLE 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \left[\begin{array}{ccc} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Notice that using the (2.9) conditions, we get $w_0(t) = L_0 t, w(t) = Lt, v(t) = L$, $L_0 = e - 1, L = e^{\frac{1}{L_0}}$. The parameters are

$$r = r_2 = 0.1210, \; r_3 = 2.9819.$$

EXAMPLE 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \; \text{ for each } \; \xi \in D.$$

Then, we get that $x^* = 0$, $w_0(t) = L_0 t, w(t) = Lt, v(t) = 1 + w_0(t)$, $L_0 = 7.5$, $L = 15$. The parameters for method are

$$r_2 = 0.0104, \; r = r_3 = 0.0040.$$

EXAMPLE 3.3. Returning back to the motivational example at the introduction of this study, we have $w_0(t) = w(t) = 96.6629073t$ and $v(t) = sup\|F(x^*)^{-1}F(x)\| = 0.7272$ for $x \in D$. Then the parameters are

$$r_2 = 0.0067, r = r_3 = 0.0006.$$

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