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Common Fixed Points of φ-Almost Contractive g-Monotone Maps in Ordered S-Metric Spaces

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Abstract In this paper, we prove the existence and uniqueness of common fixed points of g-monotone maps in partially ordered complete S-metric spaces. The results presented in this paper generalize the results of Dung, Heieu and Radojević [10] in S-metric spaces. We provide examples to demonstrate the validity of our results.

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1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions or/and the generalization of ambient spaces of the operator under consideration. Several authors generalized the ambient space in order to formulate and prove fixed points and common fixed points that generalizes well-known results from the case of usual metric spaces to the generalized metric spaces. In this direction, in 2012, Sedghi, Shobe and Aliouche [18] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. In line with the new idea of Sedghi, Shobe and Aliouche [18], some authors proved the existence of common fixed points in partially ordered S-metric spaces [9, 10, 15, 17].

In the direction of generalization of contractions, Berinde [4] defined the notion of a weak contraction mapping, which is more general than a contraction mapping. However, in [5] Berinde renamed it as an almost contraction.

Definition 1.1. [5] Let (X, d) be a metric space. A self map $T : X \to X$ is said to be an almost contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

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$d(Tx, Ty) \leq \delta d(x, y) + L\, d(y, Tx)$ for all $x, y \in X$.

Berinde [4] proved some fixed point theorems for almost contractions in a complete metric space which generalized the results of Kannan [14], Chatterjea [6] and Zamfirescu [19]. In 2008, Babu, Sandhya and Kameswari [2] introduced the class of mappings satisfying 'condition (B)' as follows:

Definition 1.2. [2] Let (X, d) be a metric space. A map $T : X \to X$ is said to satisfy 'condition (B)' if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}$$

for all $x, y \in X$.

Babu, Sandhya and Kameswari [2] proved that any map T satisfying 'condition (B)' has a unique fixed point in complete metric spaces. For more works on almost contractions and its generalizations we refer [2, 7] and the related references cited in these papers.

In 2008, Ćirić, Cakić, Rajović, Ume and Nieto [8] introduced the concept of g-monotone map and proved some common fixed point theorems for g-monotone generalized nonlinear contractions in partially ordered complete metric spaces. This result give rise to state an analogue result in partially ordered S-metric spaces. In 2014, Dung, Hieu and Radojević [10] proved an analogue of the results of Ćirić, Cakić, Rajović, Ume and Nieto [8] in partially ordered complete S-metric spaces. For more works on monotone maps we refer [9, 11, 12, 16].

Definition 1.3. [18] Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$

(S1) $S(x, y, z) \geq 0$,

(S2) $S(x, y, z) = 0$ if and only if $x = y = z$ and

(S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

The following lemmas are very useful in our subsequent discussion and in proving our main results.

Lemma 1.4. [18] In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Lemma 1.5. [10] Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$.

Definition 1.6. [18] Let (X, S) be an S-metric space. We define the following:

(i) a sequence $\{x_n\}$ in X converge to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(iii) The S-metric space (X, S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 1.7. [18] Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 1.8. [18] Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Definition 1.9. [8] Let $(X, \preceq)$ be a partially ordered set and let $F, g : X \to X$ be two maps.

(i) F is called g-nondecreasing if $gx \preceq gy$ implies $Fx \preceq Fy$ for all $x, y \in X$.

(ii) F is called g-non-increasing if $gx \preceq gy$ implies $Fy \preceq Fx$ for all $x, y \in X$.

Definition 1.10. [13] Let f and g be two self mappings of a metric space (X,d). Then f and g are said to be weakly compatible if for all $x \in X$, the equality $fx = gx$ implies $fgx = gfx$.

Definition 1.11. Let (X, S) be an S-metric space and $\preceq$, a partial order on X, then we call (X, S) a partially ordered S-metric space and we denote it by $(X, S, \preceq)$.

In the following Dung, Hieu and Radojević [10] proved the existence of common fixed points for g-nondecreasing maps in partially ordered S-metric spaces.

Theorem 1.12. [10] *Let $(X, S, \preceq)$ be a partially ordered S-metric space, $F, g : X \to X$ be two selfmaps and $\varphi : [0, \infty) \to [0, \infty)$ be a function such that*

(i) X is complete.

(ii) φ is continuous and $\varphi(t) < t$ for all $t > 0$.

(iii) $F(X) \subset g(X)$, F is a g-non-decreasing map, $g(X)$ is closed and $gx_0 \preceq Fx_0$ for some $x_0 \in X$.

(iv) For all $x, y \in X$ with $gx \preceq gy$,

$$S(Fx, Fx, Fy) \le \max\{\varphi(S(gx, gx, gy)), \varphi(S(gx, gx, Fx)),\ \varphi(S(gy, gy, Fy)), \varphi(\frac{S(gx, gx, Fy) + S(gy, gy, Fx)}{3})\}. \tag{1.1}$$

(v) If $\{gx_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty} gx_n = gz$ in $g(X)$, then $gx_n \preceq gz \preceq g(gz)$for all $n \in \mathbb{N}$.

Then F and g have a coincidence point.

Furthermore, if F and g commute at the coincidence points, then F and g have a common fixed point.

Throughout this paper, we denote
$\Phi = \{\varphi : [0, \infty) \to [0, \infty) \mid\ (i)\ \varphi$ is continuous and $(ii)\ \varphi(t) = 0$ if and only if $t = 0\}$.

We now introduce φ-almost contractive maps in S-metric spaces.

Definition 1.13. Let $(X, S, \preceq)$ be a partially ordered S-metric space and $F, g : X \to X$ be two self maps. If there exist $L \ge 0$ and $\varphi \in \Phi$ such that for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$, we have

$$S(Fx, Fy, Fz) \le M^{\varphi}_{F,g}(x, y, z)\ +\ L\ m_{F,g}(x, y, z), \tag{1.2}$$

where

$$M^{\varphi}_{F,g}(x, y, z) = \max\{\varphi(S(gx, gy, gz)), \varphi(S(gx, gx, Fx)), \varphi(S(gy, gy, Fy)), \varphi(S(gz, gz, Fz)), \varphi(\frac{S(gx, gx, Fy) + S(gy, gy, Fx)}{3}), \varphi(\frac{S(gy, gy, Fz) + S(gz, gz, Fy)}{3})\}$$

and

$$m_{F,g}(x, y, z) = L\ \min\{S(gx, gx, Fy), S(gy, gy, Fz), S(gz, gz, Fy)\},$$

then we call (F, g) satisfy φ-almost contractive condition.

Here we note that the inequality (1.1) is a spacial case of the inequality (1.2) if $L = 0$ and $gx \preceq gy$ for all $x, y \in X$.

We state the following lemma which is useful in proving our main results.

Lemma 1.14. [3] Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{1.3}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon,\ S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon \text{ and} \tag{1.4}$$

(i) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon$,

(ii) $\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \epsilon$

(iii) $\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon$.

Motivated by 'almost weak contractions' of Berinde [4, 5], 'Condition (B)' of Babu, Sandhya and Kameswari [2] in metric space setting and 'Fixed point theorems for g-monotone maps on partially ordered S-metric spaces ' of Dung, Hiew and Radojević[10], in this paper we generalize the results Dung, Hiew and Radojević [10].

In Section 2 of this paper, we prove the existence and uniqueness of common fixed points of g-monotone maps satisfying certain contractive condition by using an auxiliary function. In Section 3, we draw some corollaries from the main results and provide examples to illustrate our main results.

2. Main Results

Theorem 2.1. *Let $(X, S, \preceq)$ be a complete partially ordered S-metric space. $F, g : X \to X$ be two self maps such that F is a g-nondecreasing. Assume that*

(i) *$F(X) \subseteq g(X)$ and $g(X)$ is closed;*

(ii) *there exists $\varphi \in \Phi$ with the condition $\varphi(t) < t$ for all $t > 0$ such that the pair (F, g) satisfies φ-almost contractive condition with $L \geq 0$;*

(iii) *there exists $x_0 \in X$ such that $gx_0 \preceq Fx_0$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty} gx_n = gu$ for some $u \in X$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$.*

Then F and g have a coincidence point in X.

Furthermore, assume that

(v) F and g are weakly compatible.

Then F and g have a common fixed point in X.

Proof. By hypothesis there exists $x_0 \in X$ such that

$$gx_0 \preceq Fx_0. \tag{2.1}$$

Since $F(X) \subseteq g(X)$, there exists $x_1 \in X$ such that

$$gx_1 = Fx_0. \tag{2.2}$$

Again, since $F(X) \subseteq g(X)$, there exists $x_2 \in X$ such that

$$gx_2 = Fx_1. \tag{2.3}$$

From (2.1) and (2.2), we have

$$gx_0 \preceq gx_1. \tag{2.4}$$

Since F is a g-nondecreasing map, it follows that

$$Fx_0 \preceq Fx_1. \tag{2.5}$$

From (2.2), (2.3) and (2.5), we have

$$gx_1 \preceq gx_2. \tag{2.6}$$

Since F is a g-nondecreasing map, we get

$$Fx_1 \preceq Fx_2. \tag{2.7}$$

On continuing this process, we can constract a sequence $\{x_n\}$ in X such that

$$gx_{n+1} = Fx_n \text{ for } n = 0, 1, 2, ... \tag{2.8}$$

satisfying

$$gx_0 \preceq gx_1 \preceq ... \preceq gx_n \preceq gx_{n+1} \preceq ... \tag{2.9}$$

$$Fx_0 \preceq Fx_1 \preceq ... \preceq Fx_n \preceq Fx_{n+1} \preceq ... \,. \tag{2.10}$$

We now prove that F and g have a coincidence point. We assume that $gx_{n_0} = gx_{n_0+1}$ for some n_0. This implies x_{n_0} is the coincidence point of F and g and hence the proof is finished.

We now assume that $gx_n \preceq gx_n \preceq gx_{n+1}$ with $gx_n \neq gx_{n+1}$ for all n. Then by (1.2), we have

$$\begin{aligned}
S(Fx_n, Fx_n, Fx_{n+1}) \leq &\max\{\varphi(S(gx_n, gx_n, gx_{n+1})), \varphi(S(gx_n, gx_n, Fx_n)), \varphi(S(gx_n, gx_n, Fx_n)),\\
&\varphi(S(gx_{n+1}, gx_{n+1}, Fx_{n+1})), \varphi(\frac{S(gx_n, gx_n, Fx_n) + S(gx_n, gx_n, Fx_n)}{3}),\\
&\varphi(\frac{S(gx_{n+1}, gx_{n+1}, Fx_n) + S(gx_n, gx_n, Fx_{n+1})}{3})\} + L\min\{S(gx_n, gx_n, Fx_n),\\
&S(gx_n, gx_n, Fx_{n+1}), S(gx_{n+1}, gx_{n+1}, Fx_n)\}\\
&= \max\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)),\\
&\varphi(S(Fx_n, Fx_n, Fx_{n+1})), \varphi(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_n) + S(Fx_{n-1}, Fx_{n-1}, Fx_n)}{3}),\\
&\varphi(\frac{S(Fx_n, Fx_n, Fx_n) + S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3})\}\\
&+ L\min\{S(Fx_{n-1}, Fx_{n-1}, Fx_n), S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1}), S(Fx_n, Fx_n, Fx_n)\}\\
&= \max\{\varphi(S(Fx_{n-1}, Fx_{n-1}, Fx_n)), \varphi(S(Fx_n, Fx_n, Fx_{n+1})),\\
&\varphi(\frac{2S(Fx_{n-1}, Fx_{n-1}, Fx_n)}{3}), \varphi(\frac{S(Fx_{n-1}, Fx_{n-1}, Fx_{n+1})}{3})\}.
\end{aligned} \tag{2.11}$$

Case (i) : $\varphi(S(Fx_n, Fx_n, Fx_{n+1}))$ is the maximum.
From (2.11), we have

$$S(Fx_n, Fx_n, Fx_{n+1}) \leq \varphi(S(Fx_n, Fx_n, Fx_{n+1})) < S(Fx_n, Fx_n, Fx_{n+1}),$$

a contradiction.
Case (ii) : $\varphi(\frac{2S(Fx_{n-1},Fx_{n-1},Fx_n)}{3})$ is the maximum.
From (2.11), we have

$$S(Fx_n,Fx_n,Fx_{n+1}) \le \varphi(\frac{2S(Fx_{n-1},Fx_{n-1},Fx_n)}{3}) < \frac{2S(Fx_{n-1},Fx_{n-1},Fx_n)}{3}$$

Hence $S(Fx_n,Fx_n,Fx_{n+1}) \le S(Fx_{n-1},Fx_{n-1},Fx_n)$.
Case (iii) : $\varphi(\frac{S(Fx_{n-1},Fx_{n-1},Fx_{n+1})}{3})$ is the maximum.
From (2.11), we have

$$S(Fx_n,Fx_n,Fx_{n+1}) \le \varphi(\frac{S(Fx_{n-1},Fx_{n-1},Fx_n)}{3}) < \frac{S(Fx_{n-1},Fx_{n-1},Fx_{n+1})}{3}.$$

This implies that

$$S(Fx_n,Fx_n,Fx_{n+1}) \le \frac{1}{3}S(Fx_{n-1},Fx_{n-1},Fx_{n+1}) \le \frac{1}{3}\big(2S(Fx_{n-1},Fx_{n-1},Fx_n) + S(Fx_{n+1},Fx_{n+1},Fx_n)\big).$$

Therefore $S(Fx_n,Fx_n,Fx_{n+1}) \le \frac{2}{3}S(Fx_{n-1},Fx_{n-1},Fx_n) + \frac{1}{3}S(Fx_n,Fx_n,Fx_{n+1})$. Hence $S(Fx_n,Fx_n,Fx_{n+1}) \le S(Fx_{n-1},Fx_{n-1},Fx_n)$.
Case (iv) : $\varphi(S(Fx_{n-1},Fx_{n-1},Fx_n))$ is the maximum.
From (2.11), we have

$$S(Fx_n,Fx_n,Fx_{n+1}) \le \varphi(S(Fx_{n-1},Fx_{n-1},Fx_n)) < S(Fx_{n-1},Fx_{n-1},Fx_n).$$

Hence $S(Fx_n,Fx_n,Fx_{n+1}) \le S(Fx_{n-1},Fx_{n-1},Fx_n)$.
From all the above cases, it follows that $\{S(Fx_n,Fx_n,Fx_{n+1})\}$ is a decreasing sequence of non-negative real numbers. Hence there exists $r \ge 0$ such that

$$\lim_{n\to\infty} S(Fx_n,Fx_n,Fx_{n+1}) = r. \tag{2.12}$$

We now show that $r = 0$. Suppose that $r > 0$. On taking limits as $n \to \infty$ in (2.11) and using (2.12), we obtain

$$r \le \max\{\varphi(r),\varphi(r),\varphi(\frac{2}{3}r),\varphi(\frac{1}{3}r)\} < \max\{r,\frac{2}{3}r,\frac{1}{3}r\} = r,$$

a contradiction. Hence $r = 0$.

Let $Fx_n = y_n,\ n = 1,\ 2,\ \dots$. Now, we show that $\{y_n\}$ is a Cauchy sequence. If $\{y_n\}$ is not a Cauchy sequence, then by Lemma 1.14, there exists an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that $S(y_{m_k},y_{m_k},y_{n_k}) \ge \epsilon$, $S(y_{m_k},y_{m_k},y_{n_k-1}) < \epsilon$ and the identities (i) to (iii) of Lemma 1.14 hold.

Now, by putting $x = y = y_{m_k-1}, z = y_{n_k-1}$, and using the inequality (1.2), we have

$$\begin{aligned} S(y_{m_k},y_{m_k},y_{n_k}) = S(Fx_{m_k},Fx_{m_k},Fx_{n_k}) \le \max\{&\varphi(S(gx_{m_k},gx_{m_k},gx_{n_k})),\\ &\varphi(S(gx_{m_k},gx_{m_k},Fx_{m_k})),\varphi(S(gx_{m_k},gx_{m_k},Fx_{m_k}),\varphi(S(gx_{n_k},gx_{n_k},Fx_{n_k})),\\ &\varphi(\frac{S(gx_{m_k},gx_{m_k},Fx_{m_k})+S(gx_{m_k},gx_{m_k},Fx_{m_k})}{3}),\\ &\varphi(\frac{S(gx_{n_k},gx_{n_k},Fx_{m_k})+S(gx_{m_k},gx_{m_k},Fx_{n_k})}{3})\}\\ &+L\min\{S(gx_{m_k},gx_{m_k},Fx_{m_k}),S(gx_{m_k},gx_{m_k},Fx_{n_k}),S(gx_{n_k},gx_{n_k},Fx_{m_k})\}. \end{aligned}$$

$$= \max\{\varphi(S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{n_k-1})), \varphi(S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k})),$$
$$\varphi(S(Fx_{n_k-1}, Fx_{n_k-1}, Fx_{n_k})), \varphi(\frac{S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k}) + S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k})}{3}),$$
$$\varphi(\frac{S(Fx_{n_k-1}, Fx_{n_k-1}, Fx_{m_k}) + S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{n_k})}{3})\}$$
$$+ L \min\{S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k}), S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{n_k}), S(Fx_{n_k-1}, Fx_{n_k-1}, Fx_{m_k})\}.$$
$$= \max\{\varphi(S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{n_k-1})), \varphi(S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k})),$$
$$\varphi(S(Fx_{n_k-1}, Fx_{n_k-1}, Fx_{n_k})), \varphi(\frac{2S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{m_k})}{3}),$$
$$\varphi(\frac{S(Fx_{n_k-1}, Fx_{n_k-1}, Fx_{m_k}) + S(Fx_{m_k-1}, Fx_{m_k-1}, Fx_{n_k})}{3})\}.$$
$$= \max\{\varphi(S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})), \varphi(S(y_{m_k-1}, y_{m_k-1}, y_{m_k})),$$
$$\varphi(S(y_{n_k-1}, y_{n_k-1}, y_{n_k})), \varphi(\frac{2S(y_{m_k-1}, y_{m_k-1}, y_{m_k})}{3}),$$
$$\varphi(\frac{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k})}{3})\}.$$
$$\varphi(\frac{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k})}{3})\}$$
$$+L\min\{S(y_{m_k-1}, y_{m_k-1}, y_{m_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(y_{n_k-1}, y_{n_k-1}, y_{m_k})\}. \quad (2.13)$$

On letting $k \to \infty$ in (2.13), we have

$$\epsilon \leq \max\{\varphi(\epsilon), \varphi(0), \varphi(0), \varphi(\frac{2\epsilon}{3})\} < \max\{\epsilon, 0, 0, \frac{2\epsilon}{3}\} + L \min\{0, \epsilon, \epsilon\} = \epsilon,$$

a contradiction. Therefore $\{y_n\}$ is a Cauchy sequence. Hence $\{Fx_n\}$ and $\{gx_n\}$ are Cauchy sequences. Since $g(X)$ is closed it is complete and so there exists $u \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} Fx_n = gu. \quad (2.14)$$

We now show that u is a coincidence point of F and g. Since $gx_n \preceq gu$ for all n, by using the inequality (1.2), we have

$$S(Fx_n, Fx_n, Fu) \leq \max\{\varphi(S(gx_n, gx_n, gu)), \varphi(S(gx_n, gx_n, Fx_n)), \varphi(S(gx_n, gx_n, Fx_n)),$$
$$\varphi(S(gu, gu, Fu), \varphi(\frac{S(gx_n, gx_n, Fx_n) + S(gx_n, gx_n, Fx_n)}{3}),$$
$$\varphi(\frac{S(gu, gu, Fx_n) + S(gx_n, gx_n, Fu)}{3})\} + L \min\{S(gx_n, gx_n, Fx_n),$$
$$S(gx_n, gx_n, Fu), S(gu, gu, Fu)\}$$
$$= \max\{\varphi(S(gx_n, gx_n, gu)), \varphi(S(gx_n, gx_n, Fx_n)), \varphi(S(gx_n, gx_n, Fx_n)),$$
$$\varphi(S(gu, gu, Fu), \varphi(\frac{2S(gx_n, gx_n, Fx_n)}{3}), \varphi(\frac{S(Fx_n, Fx_n, gu) + S(gx_n, gx_n, Fu)}{3})\}$$
$$+ L \min\{S(gx_n, gx_n, gxn+1), S(gx_n, gx_n, gu), S(gu, gu, gxn+1)\}.$$

On letting $n \to \infty$, we have

$$S(gu, gu, Fu) \leq \max\{\varphi(0), \varphi(0), \varphi(0), \varphi(S(gu, gu, Fu), \varphi(\frac{S(gu, gu, Fu)}{3})\}$$
$$< \max\{0, 0, 0, S(gu, gu, Fu), \frac{S(gu, gu, Fu)}{3}\} = S(gu, gu, Fu),$$

a contradiction. Therefore $S(gu, gu, Fu) = 0$. Hence $gu = Fu$ so that u is a coincidence point of F and g.

Finally, we show that gu is a common fixed point of F and g. Since F and g are weakly compatible, we have $F(gu) = g(Fu) = g(gu)$. Since $gu \preceq g(gu)$ and if we assume $gu \neq g(gu)$, by (1.2) it follows that

$$\begin{aligned}S(Fu, Fu, F(gu)) \leq \max\{&\varphi(S(gu, gu, g(gu))), \varphi(S(gu, gu, F(u))), \varphi(S(gu, gu, F(u))),\\ &\varphi(S(g(gu), g(gu), F(gu))), \varphi(\frac{S(gu, gu, Fu) + S(gu, gu, Fu)}{3}),\\ &\varphi(\frac{S(g(gu), g(gu), Fu) + S(gu, gu, F(gu)}{3})\}.\\ &= \max\{\varphi(S(gu, gu, g(gu))), \varphi(0), \varphi(\frac{2S(g(gu), g(gu), gu)}{3})\}\\ &< \max\{S(gu, gu, g(gu)), \frac{2S(g(gu), g(gu), gu)}{3}\}\\ &= S(gu, gu, g(gu)) = S(Fu, Fu, F(gu)),\end{aligned}$$

a contradiction. Thus $\psi(S(Fu, Fu, F(gu))) = 0$. Therefore $Fu = F(gu)$. Hence $gu = Fu = F(gu) = g(gu)$. Therefore gu is a common fixed point of F and g. This completes the proof. ∎

In the following, we prove the uniqueness of common fixed point under the hypothesis 'Condition H'.

Theorem 2.2. : *In addition to the hypotheses of Theorem 2.1, we assume the following. Condition (H): for each $x, u \in X$, there exists $a \in X$ such that Fa is comparable to ga, Fx and Fu. Then F and g have a unique common fixed point in X.*

Proof. In view of the proof of Theorem 2.1, the set of common fixed points of F and g is nonempty. Let u and v be two common fixed points of F and g. i.e, $Fu = gu = u$ and $Fv = gv = v$. We now claim that $gu = gv$.

Case (i) : Fu and Fv are comparable.
If $Fu = Fv$, then we are through. We assume, without loss of generality, that $Fu \preceq Fv$ with $Fu \neq Fv$. Since $Fu = gu$ and $Fv = gv$, we have gu and gv are comparable with $gu \neq gv$. Now, by the inequality (1.2), we have

$$\begin{aligned}S(gu, gu, gv) = S(Fu, Fu, Fv)) \leq \max\{&\varphi(S(gu, gu, gv)), \varphi(S(gu, gu, Fu)), \varphi(S(gu, gu, Fu)),\\ &\varphi(S(gv, gv, Fv)), \varphi(\frac{S(gu, gu, Fu) + S(gu, gu, Fu)}{3}), \varphi(\frac{S(gu, gu, Fv) + S(gv, gv, Fu)}{3})\}\\ &+ L \min\{S(gu, gu, Fu), S(gu, gu, Fv), S(gv, gv, Fu)\}\end{aligned}$$

$$= \max\{\varphi(S(gu,gu,gv)), \varphi(S(gu,gu,gu)), \varphi(S(gv,gv,gv)), \varphi(\frac{2S(gu,gu,gu)}{3}), \varphi(\frac{2S(gu,gu,gv)}{3})\}$$
$$= \max\{\varphi(S(gu,gu,gv)), \varphi(\frac{2S(gu,gu,gv)}{3})\}$$
$$< \max\{S(gu,gu,gv), \frac{2S(gu,gu,gv)}{3}\} = S(gu,gu,gv),$$

which is a contradiction. Hence $gu = gv$, which implies that $u = v$.

Case (ii): Fu and Fv are not comparable.

In this case, by condition (H), there exists $a \in X$ such that Fa is comparable to ga, Fu and Fv. We assume, without loss of generality, that $Fu \preceq Fa$, $Fv \preceq Fa$ and $ga \preceq Fa$.

Now, we set $a = a_0$. Since $F(X) \subseteq g(X)$, there exists $a_1 \in X$ such that

$$ga_1 = Fa_0. \tag{2.15}$$

Since $Fu \preceq Fa$, $Fu = gu$ and $Fa = Fa_0 = ga_1$, we have

$$gu \preceq ga_1. \tag{2.16}$$

Since $ga_0 \preceq Fa_0 = ga_1$, we have

$$ga_0 \preceq ga_1. \tag{2.17}$$

Since F is g-nondecreasing, from (2.16) and (2.17), we get

$$Fu \preceq Fa_1, \tag{2.18}$$

and

$$Fa_0 \preceq Fa_1. \tag{2.19}$$

Since $F(X) \subseteq g(X)$, there exists $a_2 \in X$ such that

$$ga_2 = Fa_1. \tag{2.20}$$

From (2.15), (2.19) and (2.20), we have

$$ga_1 \preceq ga_2. \tag{2.21}$$

From (2.18) and (2.20), it follows that

$$gu \preceq ga_2, \text{ since } gu = Fu. \tag{2.22}$$

Since F is g-nondecreasing, from (2.21) and (2.22), we get

$$Fu \preceq Fa_2, \tag{2.23}$$

and

$$Fa_1 \preceq Fa_2. \tag{2.24}$$

On continuing this process, we can construct a sequence $\{ga_n\}$ such that

$$ga_{n+1} = Fa_n \text{ for} n = 0, 1, 2, ... \tag{2.25}$$

satisfying

$$gu \preceq ga_{n+1} \text{ and } ga_n \preceq ga_{n+1} \text{ for } n = 0, 1, 2, ... \tag{2.26}$$

Since $Fv \preceq Fa$ and $ga \preceq Fa$, by the same argument as above, it follows that

$$gv \preceq ga_{n+1} \text{and } ga_n \preceq ga_{n+1} \text{ for } n = 0, 1, 2, ... \tag{2.27}$$

By using (2.26) and applying the inequality (1.2), we have

$$\begin{aligned}
S(gu, gu, ga_{n+1}) &= S(ga_{n+1}, ga_{n+1}, gu) = S(Fa_n, Fa_n, Fu) \\
&\le \max\{\varphi(S(ga_n, ga_n, gu)), \varphi(S(ga_n, ga_n, Fa_n)), \varphi(S(ga_n, ga_n, Fa_n)), \\
&\qquad \varphi(S(gu, gu, Fu)), \varphi(\frac{S(ga_n, ga_n, Fa_n) + S(ga_n, ga_n, Fa_n)}{3}), \\
&\qquad \varphi(\frac{S(ga_n, ga_n, Fu) + S(gu, gu, Fa_n)}{3})\} \\
&\quad + L \min\{S(ga_n, ga_n, Fa_n), S(ga_n, ga_n, Fu), S(gu, gu, Fa_n)\} \\
&= \max\{\varphi(S(ga_n, ga_n, gu)), \varphi(S(ga_n, ga_n, ga_{n+1})), \varphi(S(gu, gu, gu)), \\
&\qquad \varphi(\frac{2S(ga_n, ga_n, ga_{n+1})}{3}), \varphi(\frac{S(ga_n, ga_n, gu) + S(gu, gu, ga_{n+1})}{3})\} \\
&\quad + L \min\{S(ga_n, ga_n, ga_{n+1}), S(ga_n, ga_n, gu), S(gu, gu, ga_{n+1})\}. \qquad (2.28)
\end{aligned}$$

Since $ga_n \preceq ga_{n+1}$, by using the inequality (1.2), it is easy to see that $\{ga_n\}$ is Cauchy as in the proof of Theorem 2.1. Since $g(X)$ is closed and hence complete, there exists $w \in X$ such that $ga_n \to gw$ as $n \to \infty$.

We now, show that $gw = gu$. Suppose that $gw \neq gu$. On letting $n \to \infty$ in (2.28), we have

$$\begin{aligned}
S(gu, gu, gw) = S(gw, gw, gu) &\le \max\{\varphi(S(gw, gw, gu)), \varphi(\frac{2S(gw, gw, gu)}{3}\} \\
&< \max\{S(gw, gw, gu), \frac{2S(gw, gw, gu)}{3}\} = S(gw, gw, gu),
\end{aligned}$$

a contradiction. Hence $S(gw, gw, gu) = 0$ so that

$$gw = gu. \qquad (2.29)$$

We now show that $gw = gv$. If possible assume that $gw \neq gv$. By (2.27), since $gv \preceq gv \preceq ga_{n+1}$, and by applying the inequality (1.2), we have

$$\begin{aligned}
S(gv, gv, ga_{n+1}) &= S(ga_{n+1}, ga_{n+1}, gv) = S(Fa_n, Fa_n, Fv)) \\
&\le \max\{\varphi(S(ga_n, ga_n, gv)), \varphi(S(ga_n, ga_n, Fa_n)), \varphi(S(ga_n, ga_n, Fa_n)), \\
&\qquad \varphi(S(gv, gv, Fv)), \varphi(\frac{S(ga_n, ga_n, Fa_n) + S(ga_n, ga_n, Fa_n)}{3}), \\
&\qquad \varphi(\frac{S(ga_n, ga_n, Fu) + S(gv, gv, Fa_n)}{3})\} \\
&\quad + L \min\{S(ga_n, ga_n, ga_n), S(ga_n, ga_n, Fv), S(gv, gv, Fa_n)\}. \\
&= \max\{\varphi(S(ga_n, ga_n, gv)), \varphi(S(ga_n, ga_n, ga_{n+1})), \varphi(S(gv, gv, gv)), \\
&\qquad \varphi(\frac{S(ga_n, ga_n, gv) + S(gv, gv, ga_{n+1})}{3})\} \\
&\quad + L \min\{S(ga_n, ga_n, ga_{n+1}), S(ga_n, ga_n, Fv), S(gv, gv, ga_{n+1})\}. \qquad (2.30)
\end{aligned}$$

On letting $n \to \infty$ in (2.30), we have

$$S(gw, gw, gv) \leq \max\{\varphi(S(gw, gw, gv)), \varphi(\frac{2S(gw, gw, gv)}{3}\}$$

$$< \max\{S(gw, gw, gv), \frac{2S(gw, gw, gv)}{3}\} = S(gw, gw, gv),$$

a contradiction. Hence $S(gw, gw, gv) = 0$. This implies that $\lim_{n\to\infty} ga_n = gv$ so that

$$gw = gv. \tag{2.31}$$

Hence from (2.29) and (2.31), we get $gu = gv$. This completes the proof. ∎

3. Corollaries and Examples

If $L = 0$ in Theorem 2.1, we have the following.

Corollary 3.1. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and $F, g : X \to X$ be two maps such that F is a g-non-decreasing map. Assume that*

(i) $F(X) \subseteq g(X)$ and $g(X)$ is closed;

(ii) there exists $\varphi \in \Phi$ with the condition $\varphi(t) < t$ for all $t > 0$ such that for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$

$$S(Fx, Fy, Fz) \leq \max\{\varphi(S(gx, gy, gz)), \varphi(S(gx, gx, Fx)), \varphi(S(gy, gy, Fy)), \varphi(S(gz, gz, Fz)),$$
$$\varphi(\frac{S(gx, gx, Fy) + S(gy, gy, Fx)}{3}), \varphi(\frac{S(gy, gy, Fz) + S(gz, gz, Fy)}{3})\};$$

(iii) there exists $x_0 \in X$ such that $gx_0 \preceq Fx_0$;

(iv) if $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is a non-decreasing sequence with
$$\lim_{n\to\infty} gx_n = gu \ for \ some \ u \in X, \ then \ gx_n \preceq gu \preceq g(gu) \ for \ all \ n \in \mathbb{N}.$$

Then F and g have a coincidence point in X.

Furthermore, assume that

(v) F and g are weakly compatible.

Then F and g have a common fixed point in X.

Remark 3.2. If $L = 0$ and $x = y$ in the inequality (1.2), we get Theorem 2.5 of Dung, Hieu and Radojević [10] which is noted here as Theorem 1.12.

If g is the identity map on X in Theorem 2.1, we have the following.

Corollary 3.3. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and $F : X \to X$ be a map such that F is a non-decreasing map. Assume that*

(i) there exist $L \geq 0$ and $\varphi \in \Phi$ with the condition $\varphi(t) < t$ for all $t > 0$ such that for all $x, y, z \in X$ with $x \preceq y \preceq z$

$$S(Fx, Fy, Fz) \leq \max\{\varphi(S(x, y, z)), \varphi(S(x, x, Fx)), \varphi(S(y, y, Fy)), \varphi(S(z, z, Fz)),$$
$$\varphi(\frac{S(x, x, Fy) + S(y, y, Fx)}{3}), \varphi(\frac{S(y, y, Fz) + S(z, z, Fy)}{3})\}$$
$$+L \min\{S(x, x, Fy), S(y, y, Fz), S(z, z, Fy)\};$$

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$;

(iii) if $\{x_n\}$ is a non decreasing sequence in X with $\lim_{n\to\infty} x_n = u$ for some $u \in X$,

then $x_n \preceq u$ for all $n \in \mathbb{N}$. Then F has a fixed point in X.

Corollary 3.4. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and $F : X \to X$ be non-decreasing map. Assume that*

(i) there exists $k \in [0, 1)$ such that for all $x, y, z \in X$ with $x \preceq y \preceq z$

$$S(Fx, Fy, Fz) \le k \max\Big\{S(x, y, z), S(x, x, Fx), S(y, y, Fy), S(z, z, Fz), \frac{S(x, x, Fy) + S(y, y, Fx)}{3}, \frac{S(y, y, Fz) + S(z, z, Fy)}{3}\Big\};$$

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$;

(iii) if $\{x_n\}$ is a non-decreasing sequence with $\lim_{n\to\infty} x_n = u$ in X, then $x_n \preceq u$ for all $n \in \mathbb{N}$.

Then F has a fixed point.

Proof. Follows by choosing $\varphi(t) = kt,\ t \ge 0$ and $L = 0$ in Corollary 3.3. ∎

If $L = 0$ in Corollary 3.3, we have the following.

Corollary 3.5. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and $F : X \to X$ be a map such that F is a non-decreasing map. Assume that*

(i) there exists $\varphi \in \Phi$ with the condition $\varphi(t) < t$ for all $t > 0$ such that for all $x, y, z \in X$ with $x \preceq y \preceq z$

$$S(Fx, Fy, Fz) \le \max\Big\{\varphi(S(x, y, z)), \varphi(S(x, x, Fx)), \varphi(S(y, y, Fy)), \varphi(S(z, z, Fz)), \varphi\Big(\frac{S(x, x, Fy) + S(y, y, Fx)}{3}\Big), \varphi\Big(\frac{S(y, y, Fz) + S(z, z, Fy)}{3}\Big)\Big\};$$

(ii) there exists $x_0 \in X$ such that $x_0 \preceq Fx_0$;

(iii) if $\{x_n\}$ is a non decreasing sequence in X with $\lim_{n\to\infty} x_n = u$ for some $u \in X$, then $x_n \preceq u$ for all $n \in \mathbb{N}$.

Then F has a fixed point in X.

In the following we provide examples in support of our main results. For simplicity, we write

$$M^{\varphi}_{F,g}(x, y, z) = \max\Big\{\varphi(S(gx, gy, gz)), \varphi(S(gx, gx, Fx)), \varphi(S(gy, gy, Fy)), \varphi(S(gz, gz, Fz)), \varphi\Big(\frac{S(gx, gx, Fy) + S(gy, gy, Fx)}{3}\Big), \varphi\Big(\frac{S(gy, gy, Fz) + S(gz, gz, Fy)}{3}\Big)\Big\} \text{ and}$$

$$m_{F,g}(x, y, z) = \min\{S(gx, gx, Fy), S(gy, gy, Fz), S(gz, gz, Fy)\}.$$

The following example is in support of Theorem 2.1.

Example 3.6. Let $X = [0, \frac{3}{4}]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \max\{\,|\,x - z\,|, |\,y - z\,|\,\}.$$

We define an order relation $\preceq$ on X by

$$\preceq := \{(x, x) : x \in X\} \cup \{(\frac{1}{2}, 0), (\frac{1}{4}, 0), (\frac{3}{8}, 0), (\frac{3}{16}, 0), (\frac{1}{4}, \frac{3}{16}), (\frac{1}{2}, \frac{3}{8})\},$$

where $x \preceq y \iff x \ge y$. Then $(X, S, \preceq)$ is a partially ordered complete S-metric space.

We define $F, g : X \to X$ by

$$F(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{3}{4} & \text{if } x \in (\frac{1}{2}, \frac{3}{4}] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}] \\ 2x - \frac{3}{4} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}]. \end{cases}$$

Clearly F is a g-nondecreasing map and g is continuous, $F(X) \subseteq g(X)$ and $g(X)$ is closed. We define $\varphi : [0, \infty) \to [0, \infty)$ by

$$\varphi(t) = \frac{1}{2}t \quad \text{if } t \geq 0.$$

In the following, we show that the inequality (1.2) holds with $L = 2$. For this purpose, we consider the cases for which $gx \preceq gy \preceq gz$.

Case (i): $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, 0)$.

In this case, we have

$$\begin{aligned} S(Fx, Fy, Fz) = S(F\frac{1}{2}, F\frac{1}{2}, F0) &= |\frac{1}{2} - 0| = \frac{1}{2} < \frac{5}{8} \\ &= \varphi(S(g\frac{1}{2}, g\frac{1}{2}, g0)) + 2S(g\frac{1}{2}, g\frac{1}{2}, F\frac{1}{2}) \\ &\leq M^{\varphi}_{F,g}(x, y, z) + L\, m_{F,g}(x, y, z). \end{aligned}$$

Case (ii): $(x, y, z) = (\frac{3}{8}, \frac{3}{8}, 0)$.

In this case, we have

$$\begin{aligned} S(Fx, Fy, Fz) = S(F\frac{3}{8}, F\frac{3}{8}, F0) &= |\frac{3}{8} - 0| = \frac{3}{8} < \frac{3}{4} \\ &= \varphi(S(g\frac{3}{8}, g\frac{3}{8}, g0)) + 2S(g\frac{3}{8}, g\frac{3}{8}, F\frac{3}{8}) \\ &\leq M^{\varphi}_{F,g}(x, y, z) + L\, m_{F,g}(x, y, z). \end{aligned}$$

Case (iii): $(x, y, z) = (\frac{1}{2}, \frac{3}{8}, 0)$.

In this case, we have

$$\begin{aligned} S(Fx, Fy, Fz) = S(F\frac{1}{2}, F\frac{3}{8}, F0) &= \max\{|\frac{1}{2} - 0|, |\frac{3}{8} - 0|\} = \frac{1}{2} < \frac{5}{8} \\ &= \varphi(S(g\frac{1}{2}, g\frac{3}{8}, g0)) + 2S(g\frac{3}{8}, g\frac{3}{8}, F\frac{3}{8}) \\ &\leq M^{\varphi}_{F,g}(x, y, z) + L\, m_{F,g}(x, y, z). \end{aligned}$$

Case (iv): $(x, y, z) = (\frac{1}{2}, \frac{1}{2}, \frac{3}{8})$.

In this case, we have

$$\begin{aligned} S(Fx, Fy, Fz) = S(F\frac{1}{2}, F\frac{1}{2}, F\frac{3}{8}) &= |\frac{1}{2} - \frac{3}{8}| = \frac{1}{8} < \frac{5}{16} \\ &= \varphi(S(g\frac{1}{2}, g\frac{1}{2}, g\frac{3}{8})) + 2S(g\frac{1}{2}, g\frac{1}{2}, F\frac{3}{8}) \\ &\leq M^{\varphi}_{F,g}(x, y, z) + L\, m_{F,g}(x, y, z). \end{aligned}$$

Hence the inequality (1.2) holds.

Now, we choose $x_0 = \frac{3}{4}$. Then $g(\frac{3}{4}) \preceq F(\frac{3}{4})$. Also, if $\{gx_n\}$ is a non-decreasing sequence such that $\lim_{n\to\infty} gx_n = gx \in X$, then $gx_n \preceq gx$ for all $n \in \mathbb{N}$. Further, it is easy to see that F and g are weakly compatible.

Therefore F, g and φ satisfy all the hypotheses of Theorem 2.1, and F and g have two common fixed points 0 and 2.

Here we observe that for 0, $\frac{3}{4} \in X$, $F(0)$ and $F(\frac{3}{4})$ are not comparable. But there does not exist $a \in X$ such that $F(a)$ is comparable to ga, $F(0)$ and $F(\frac{3}{4})$. i.e., Condition (H) of the hypothesis of Theorem 2.2 fails to hold and we observe that F and g have more than one common fixed points.

We now show that the inequality (1.2) fails to hold if $L = 0$. For this purpose, we choose $x = y = \frac{3}{8}$ and $z = 0$, and since $g\frac{3}{8} \preceq g0$, we have

$$S(Fx, Fx, Fz) = S(F\frac{3}{8}, F\frac{3}{8}, F0) = \frac{3}{8} \npreceq \frac{3}{32} = \max\{\varphi(S(g\frac{3}{8}, g\frac{3}{8}, g0)), \varphi(S(g\frac{3}{8}, g\frac{3}{8}, F\frac{3}{8})),$$

$$\varphi(S(g0, g0, F0)), \varphi(\frac{S(g\frac{3}{8}, g\frac{3}{8}, F0) + S(g0, g0, F\frac{3}{8})}{3})\} = M^{\varphi}_{F,g}(x, x, z).$$

for any $\varphi \in \Phi$, which shows the importance of L in Theorem 2.1. Consequently, the inequality (1.1) fails to hold for any $\varphi \in \Phi$ so that Theorem 1.12 is not applicable.

Hence Remark 3.2 and Example 3.6 suggest that Theorem 2.1 is a generalization of Theorem 1.12.

The following example is in support of Theorem 2.2.

Example 3.7. Let $X = \{\frac{1}{2^{n+1}} : n = 0, 1, 2, ...\} \cup \{0, 1, 2\}$. We define a partial order $\preceq$ on X by

$$\preceq := \{(x, x) : x \in X\} \cup \{(1, 0), (1, \frac{1}{2^{n+1}}), (2, 0), \} \cup \{(\frac{1}{2^{n+1}}, 0)\},$$

where $x \preceq y \iff x \geq y$ in the usual sense.

We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = |x - z| + |y - z|$$

Then $(X, S, \preceq)$ is a partially ordered complete S-metric space.

We define $F,\ g : X \to X$ by

$$F = \begin{pmatrix} 0 & 1 & 2 & \frac{1}{2^{n+1}} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2^{n+2}} \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 & 1 & 2 & \frac{1}{2^{n+1}} \\ 0 & \frac{1}{4} & 2 & \frac{1}{2^{n+1}} \end{pmatrix} \text{ for } n = 0, 1, 2, 3, \ldots .$$

It is clear that F is g-nondecreasing on X, $F(X) \subseteq g(X)$ and $g(X)$ is closed.
We choose $x_0 = 1$. Then $g(x_0) \preceq F(x_0)$ holds.

We define $\varphi : [0, \infty) \to [0, \infty)$ by

$$\varphi(t) = \begin{cases} -\frac{1}{2}(t-1)^2 + \frac{1}{2} & \text{if } t \in [0, 1] \\ t - \frac{1}{2} & \text{if } t \geq 1. \end{cases}$$

We now consider nontrivial cases such that $gx \preceq gy \preceq gz$ and show that the inequality (1.2) holds with $L = 2$.

Case (i): $(x, y, z) = (1, 1, 0)$.

In this case, we have

$$S(Fx, Fy, Fz) = S(F1, F1, F0) = 1 < \frac{11}{8} = \varphi(S(g1, g1, g0)) + 2S(g1, g1, F1)$$
$$\leq M^{\varphi}_{F,g}(x, y, z) + L\ m_{F,g}(x, y, z).$$

Case (ii): $(x, y, z) = (2, 2, 0)$.
In this case, we have

$$S(Fx, Fy, Fz) = S(F2, F2, F0) = 2 < \frac{7}{2} = \varphi(S(g2, g2, g0)) \leq M^{\varphi}_{F,g}(x, y, z) + L\, m_{F,g}(x, y, z).$$

Case (iii): $g\frac{1}{2^{n+1}} \preceq g\frac{1}{2^{n+1}} \preceq g0$ where $n = 0, 1, 2, \ldots$.
In this case,

$$\begin{aligned} S(Fx, Fy, Fz) &= S(F\frac{1}{2^{n+1}}, F\frac{1}{2^{n+1}}, F0) = S(\frac{1}{2^{n+2}}, \frac{1}{2^{n+2}}, 0) = \frac{1}{2^{n+1}} \\ &\leq -\frac{1}{2}(1 - \frac{1}{2^n})^2 + \frac{1}{2} = \varphi(S(g\frac{1}{2^{n+1}}, g\frac{1}{2^{n+1}}, g0)) \\ &\leq M^{\varphi}_{F,g}(x, y, z) + L\, m_{F,g}(x, y, z). \end{aligned}$$

From all the above cases, it follows that the inequality (1.2) holds.

Here we observe that $F1$ and $F\frac{1}{2^{n+1}}$ are not comparable. But there exists $0 \in X$ such that $F0$ is comparable to $F1$, $F\frac{1}{2^{n+1}}$ and $g0$. In fact for each $x, u \in X$ there exists $0 \in X$ such that $F0$ is comparable to Fx, Fu and $g0$. Therefore 'Condition (H)' holds.

Hence F, g and φ satisfy all the hypotheses of Theorem 2.2 and 0 is the unique common fixed point of F and g.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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