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# COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACE

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**Abstract** In this paper, we prove a common fixed point theorem for weakly compatible mappings in intutionistic fuzzy metric space. we use (S-B) property in intutionistic fuzzy metric space and replace the completeness of the space with a set of alternative conditions. Our result improve some earlier results.

**MSC:** 47H10 , 54H25.

**Keywords:** Common fixed point, Intutionistic fuzzy metric space, (S-B) property, Weakly Compatible mappings.

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## 1. Introduction

It is of no doubt that the concept of fuzzy set given by Zadeh [29] helped other researchers to introduce and develop the fuzzy metric in distinct forms. After a decade, Kramosil and Michelak [17] introduced the concept of fuzzy metric space which can be regarded as the generalization of statatistical (probabilistic) metric space, thereby opened a road map to explore and analyze further in fuzzy metric spaces. Credit for further development goes to Grabiec [9] who extended two fixed point theorems of Banach [5] and Edelstein [8] for contractive mappings of complete and compact fuzzy metric space in the sense of Kramosil and Michalek [17]. Thereafter, George and Veeramani ([10],[11]) modified the concept of fuzzy metric space given by Kramosil and Michalek [17] and defined a Hausdroff topology on it.
Attansov ([3],[4]) genarilized the fuzzy meteric space to intutuionistic fuzzy metric space and proved some common fixed point theorems for the same. In 1994, Mishra et al. [20] extended the notion of compatible maps introduced by Jungck et al. [12] in metric space under the name of asymptoticaly commuting maps to fuzzy metric spaces. Park



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[21] in 2004, using the idea of intuitionistic fuzzy sets defined the notion of intutionistic fuzzy metric space with the help of continuous t-norm and t-conorm as a generalization of fuzzy metric space due to Georage and Veermani ([10],[11]). Sharma et.al.([25, [26]) introduced and studied the concept of common fixed point for weakly and multivalued mappings in intuitionistic fuzzy metric spaces . For definitions and preliminaries, we refer to ( Alaca et al. [1], Anderson et al.[2], Deng [6], Erceg [7], Kaleva et al. [16], Kubiaczyk and Sharma [18], Jungck ([13], [14]),Jungck and Rhoades [15], Schweizer and Skaler[22], Sessa[23], Sharma and Bamboria [24], Sharma and Deshpande [27] and Turkoglu et al.[28]) respectively.

## 2. Preliminaries

**Definition.**[22] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a, \forall a \in [0,1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d, \forall a, b, c, d \in [0,1]$.

**Definition.** [22] A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $\Diamond$ is satisfying the following conditions:

(i) $\Diamond$ is commutative and associative,

(ii) $\Diamond$ is continuous,

(iii) $a \Diamond 0 = a, \forall a \in [0,1]$,

(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d, \forall a, b, c, d \in [0,1]$.

**Remark** 1.The concept of triangular norms (t norms) and triangular conorms (t-conorms) are known as the axiomatic sekeltons that we use for characterizing fuzzy intersections and unions respectively. These concepts were originally introduced by Menger[19] in his study of Statistical metric spaces.

**Definition.** [1] A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric spaces if $X$ is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and $M, N$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:
$\forall x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, t) + N(x, y, t) \leq 1$,

(ii) $M(x, y, 0) = 0$,

 

(iii) $M(x, y, t) = 1, \forall t > 0$ if and only if $x = y$,

(iv) $M(x, y, t) = M(y, x, t)$,

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(vi) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(vii) $lim_{t\to\infty} M(x, y, t) = 1, \forall x, y \in X$,

(viii) $N(x, y, 0) = 1$,

(ix) $N(x, y, t) = 0, \forall t > 0$ if and only if $x = y$,

(x) $N(x, y, t) = N(y, x, t)$,

(xi) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$,

(xii) $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous,

Then (M, N) is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

**Remark** 2. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated , i.e., $x\Diamond y = 1 - ((1 - x) * (1 - y)), \forall x, y \in X$.

**Example.** Let $(X, d)$ be a metric space. Define t-norm $a*b = min\{a, b\}$ and t-conorm $a\Diamond b = max\{a, b\}$ and $\forall x, y \in X$ and $t > 0, M_d(x, y, t) = \frac{t}{t+d(x,y)}, N_d(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric (M, N) induced by the metric d, the standard intuitionistic fuzzy metric.

**Example.** [1] Let $X = N$. Define $a*b = max\{0, a+b-1\}$ and $a\Diamond b = a+b-ab, \forall a, b \in [0, 1]$ and let M and N be the fuzzy sets on $X^2 \times (0, \infty)$ as follows :
$\forall x, y \in X and t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

$$M(x, y, t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } y \leq x \end{cases}$$

and

$$N(x, y, t) = \begin{cases} \frac{y-x}{y}, & \text{if } x \leq y \\ \frac{x-y}{x}, & \text{if } y \leq x \end{cases}$$

**Remark** 3. In intuitionistic fuzzy metric space $X, M(x, y, .)$ is non- decreasing and $N(x, y, .)$ is non-increasing, $\forall x, y \in X$.

**Definition.** [1] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ (denoted by $lim_{n\to\infty}\{x_n\} = x)\, if, \forall t > 0$,
$lim_{n\to\infty} M(x_n, x, t) = 1,\ lim_{n\to\infty} N(x_n, x, t) = 0.$

(ii) A sequence $\{x_n\}$ in $X$ is said to be a cauchy sequence if $\forall\, t > 0$ and $p > 0$
$lim_{n\to\infty} M(x_{n+p}, x, t) = 1,\ lim_{n\to\infty} N(x_{n+p}, x, t) = 0.$

**Definition.** [1] An intutionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be complete if and only if every cauchy sequence in $X$ is convergent. It is called compact if every sequence contains a convergent subsequence.

**Remark** 4. Since $*$ and $\Diamond$ are continuous, the limit is uniquely determined from Definition [1]((v) and (xi)), respectively.

**Lemma 1.** Let $(X, M, N, *, \Diamond)$ be an intutionistic fuzzy metric space and $\{y_n\}$ be sequence in $X$. If there exists a number $k \in (0, 1)$ such that

(i) $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, kt)$

(ii) $N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, kt), \forall t > 0$ and $n = 1, 2, ...,$ then $\{y_n\}$ is a cauchy sequence in $X$.

*Proof.* By simple induction with the condition (i) with the help of Alaca et. al. [1], we have $\forall t > 0$ and $n = 1, 2, ...,$

$(iii)\ M(y_{n+1}, y_{n+2}, t) \geq M(y_1, y_2, \frac{t}{k^n}), N(y_{n+1}, y_{n+2}, t) \leq N(y_1, y_2, \frac{t}{k^n})$

Thus by (iii) and definition (1.3), (v) and (xi), for any positive integer $p$ and real number $t > 0$, we have

$M(y_{n+p}, y_{n+1}, t) \geq M(y_n, y_{n+1}, \frac{t}{p}) * ...p - times... * M(y_{n+p-1}, y_{n+p}, \frac{t}{p})$

$\geq M(y_1, y_2, \frac{t}{pk^{n-1}}) * ...p - times... * M(y_1, y_2, \frac{t}{pk^{n+p-1}})$
and

$N(y_{n+p}, y_{n+1}, t) \leq N(y_n, y_{n+1}, \frac{t}{p}) \Diamond ...p - times... \Diamond N(y_{n+p-1}, y_{n+p}, \frac{t}{p})$

$\leq N(y_1, y_2, \frac{t}{pk^{n-1}}) \Diamond ...p - times... \Diamond N(y_1, y_2, \frac{t}{pk^{n+p-1}}).$

Therefore by Definition [1]((vii) and (xiii)), we have

$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * ...p - times... * 1 \geq 1,$
and

$\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 0 \Diamond ...p - times... \Diamond 0 \leq 0$, which implies that $\{y_n\}$ is a cauchy sequence in $X$. This completes the proof. The following Lemma establishes a relationship between $x$ and $y$ by virtue of Kubiaczyk and Sharma[18]

**Lemma 2.** Let $(X, M, N, *, \diamondsuit)$ be an intutionistic fuzzy metric space and $\forall x, y \in X, t > 0$ and if for a number $k \in (0,1), M(x,y,kt) \geq M(x,y,t)$ and $N(x,y,kt) \leq N(x,y,t)$, then $x = y$

*Proof.* Since $M(x,y,kt) \geq M(x,y,t)$ and $N(x,y,t) \leq N(x,y,t)$, then using results of Kubiaczyk and Sharma[18], we have

$M(x,y,kt) \geq M(x,y,\frac{t}{k})$ and $N(x,y,t) \leq N(x,y,\frac{t}{k})$ .
By repeated application of above inequalities , we have
$M(x,y,t) \geq M(x,y,\frac{t}{k})$

$M(x,y,kt) \geq M(x,y,\frac{t}{k^2}) \geq ... \geq M(x,y,\frac{t}{k^n}) \geq ...,$
and
$N(x,y,kt) \leq N(x,y,\frac{t}{k^2}) \leq ... \leq N(x,y,\frac{t}{k^n}) \leq ..., n \in N$ which tend to 1 and 0, respectively as $n \to \infty$. Thus $M(x,y,t) = 1$ and $N(x,y,t) = 0$
for all $t > 0$ and we get $x = y$.

**Definition.** [1] Let $A$ and $B$ be maps from an intutionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ into itself. The maps $A$ and $B$ are said to be compatible if $\forall t \geq 0$,

$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ and $\lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

**Definition.** [15] Two self maps $A$ and $B$ on a set $X$ are said to be weakly compatible if they commute at coincidence points; i.e., if $Au = Bu$ for some $u \in X$, then $ABu = BAu$.

**Definition.** [24] Let $S$ and $T$ be two self mappings of a fuzzy metric space $(X, M, *)$ we say that $S$ and $T$ satisfy the property (S-B) if there exists a sequence $\{x_n\}$ in $X$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

**Example.** [24]Let $X = [0, +\infty)$. Define $S, T : X \to X$ by $Tx = \frac{x}{4}$ and $Sx = \frac{3x}{4}, \forall x \in X$. Consider the sequence $\{x_n\} = \frac{1}{n}$, clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then $S$ and $T$ satisfy (S-B).

**Example.** [24] Let $X = [2, +\infty)$. Define $S, T : X \to X$ by $Tx = x+1$ and $Sx = 2x+1$, $\forall x \in X$. Suppose property (S-B) holds; then there exists in $X$ a sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore $\lim_{n\to\infty} x_n = z - 1$ and $\lim_{n\to\infty} x_n = \frac{(z-1)}{2}$. Then $z = 1$, which is a contradiction. Since $1 \notin X$.
Hence $S$ and $T$ do not satisfy the property (S-B).

**Remark** 5. It is clear from the definition of Mishra et.al. [20] and Sharma and Deshpande [27] that two self mappings $S$ and $T$ of a fuzzy metric space $(X, M, *)$ will be noncompatible if there exists at least one sequence $\{x_n\} \in X$ such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = z$ for some $z \in X$, but $\lim_{n\to\infty} M(STx_n, TSx_n, t)$ is either not equal to 1 or non-existant. Therefore two non-compatible self mappings of fuzzy metric space $(X, M, *)$ satisfy the property (S-B).

 

## 3. Main Results

The following is our main result:

**Theorem 1.** *Let* $(X, M, N, *, \diamondsuit)$ *be an intutionistic fuzzy metric space with continuous* $t-$ *norm* $*$ *and continuous t-conorm* $\diamondsuit$ *defined by* $t*t \geq t$ *and* $(1-t)\diamondsuit(1-t) \leq (1-t), \forall t \in [0,1]$ *and let* $A, B, S, T, I, J, L, U, P$ *and* $Q$ *be mappings from* $X$ *into itself such that*

(i) $P(X) \subset ABIL(X), Q(X) \subset STJU(X)$,
(ii) $\{P, STJU\}$ *or* $\{Q, ABIL\}$ satisfies the property $(S-B)$,
there exists a constant $k \in (0,1)$ such that
(iii)$M(Px, Qy, kt) \geq M(ABILy, STJUx, t) * M(Px, STJUx, t) * M(Qy, ABILy, t) * M(Qy, STJUx, t) * M(Px, ABILy, t)$
and
$N(Px, Qy, kt) \leq N(ABILy, STJUx, t)\diamondsuit N(Px, STJUx, t)\diamondsuit N(Qy, ABILy, t)\diamondsuit N(Qy, STJUx, t)$ $\diamondsuit N(Px, ABILy, t)$, $\forall x, y \in X$ and $t > 0$
(iv) if one of $P(X), ABIL(X), STJU(X), Q(X)$ is a closed subspace of $X$, then
(v) $P$ and $STJU$ have a coincidence point and
(vi) $Q$ and $ABIL$ have a coincidence point .
Further, if
(vii) $AB = BA, AI = IA, AL = LA, BI = IB, BL = LB, IL = LI, QL = LQ, QI = IQ, QB = BQ, ST = TS, SJ = JS, SU = US, TJ = JT, TU = UT, JU = UJ, PU = UP, PJ = JP, PT = TP$,
(viii) the pairs $\{P, STJU\}$ and $\{Q, ABIL\}$ are weakly compatible, then
(ix) $A, B, S, T, I, J, L, U, P$ and $Q$ have a unique common fixed point in $X$.

*Proof.* Suppose that $\{Q, ABIL\}$ satisfies the property $(S-B)$. Then there exists a sequence $\{x_n\}$ in $X$ such that $lim_{n\to\infty}Qx_n = lim_{n\to\infty}ABILx_n = z$, for some $z \in X$. Since $Q(X) \subset STJU(X)$ , there exists in $X$ a sequence $\{y_n\}$ such that $Qx_n = STJUy_n$. Hence $lim_{n\to\infty}STJUy_n = z$. Let us show that $lim_{n\to}Py_n = z$. Suppose for some $r \in X$, $lim_{n\to\infty}Py_n = r$, where $r \neq z$.

Indeed in view of (iii), we have

$M(Py_n, Qx_n, kt) \geq M(ABILx_n, STJUy_n, t)*M(Py_n, STJUy_n, t)*M(Qx_n, ABILx_n, t)*$

$M(Qx_n, STJUy_n, t) * M(Py_n, ABILx_n, t)$,

$M(Py_n, Qx_n, kt) \geq M(ABILx_n, Qx_n, t)*M(Py_n, Qx_n, t)*M(Qx_n, ABILx_n, t)*M(Qx_n, Qx_n, t)*$
$M(Py_n, ABILx_n, t)$ ,
$M(Py_n, Qx_n, kt) \geq 1 * M(Py_n, Qx_n, t) * 1 * 1 * M(Py_n, ABILx_n, t)$. Letting $n \to \infty$ , we have

$M(r, z, kt) \geq M(r, z, t)$. By Lemma (2), we have $r = z$. Therefore we deduce that $lim_{n\to\infty}Py_n = z$.
On the other hand

$N(Py_n, Qx_n, kt) \leq N(ABILx_n, STJUy_n, t)\Diamond N(Py_n, STJUy_n, t)\Diamond N(Qx_n, ABILx_n, t)\Diamond N(Qx_n, STJUy_n, t)\Diamond N(Py_n, ABILx_n, t),$

$N(Py_n, Qx_n, kt) \leq N(ABILx_n, Qx_n, t)\Diamond N(Py_n, Qx_n, t)\Diamond N(Qx_n, ABILx_n, t)\Diamond N(Qx_n, Qx_n, t)\Diamond N(Py_n, ABILx_n, t),$

$N(Py_n, Qx_n, kt) \leq 0\Diamond N(Py_n, Qx_n, t)\Diamond 0\Diamond 0\Diamond N(Py_n, ABILx_n, t).$

Letting $n \to \infty$ , we have $N(r, z, kt) \leq N(r, z, t)$. By Lemma (2) we have, $r = z$. Therefore we deduce that $lim_{n\to\infty} Py_n = z$.

Suppose that $STJU(X)$ is a closed subset of $X$. Then $z = STJUw$ for some $w \in X$. Subsequently, we have

$lim_{n\to\infty} Py_n = lim_{n\to\infty} Qx_n = lim_{n\to\infty} ABILx_n = lim_{n\to\infty} STJUy_n = STJUw$, by (iii), we have

$M(Pw, Qx_n, kt) \geq M(ABILx_n, STJUw, t)*M(Pw, STJUw, t)*M(Qx_n, ABILx_n, t)*$

$M(Qx_n, STJUw, t) * M(Pw, ABILx_n, t).$

Letting $n \to \infty$ , we have

$M(Pw, z, kt) \geq M(z, z, t) * M(Pw, z, t) * M(z, z, t) * M(z, z, t) * M(Pw, z, t),$

$M(Pw, z, kt) \geq M(Pw, z, t)$ and by (iii),we have

$N(Pw, Qx_n, kt) \leq N(ABILx_n, STJUw, t)\Diamond N(Pw, STJUw, t)\Diamond N(Qx_n, ABILx_n, t) \Diamond N(Qx_n, STJUw, t)\Diamond N(Pw, ABILx_n, t).$

Letting $n \to \infty$ , we have

$N(Pw, z, kt) \leq N(z, z, t)\Diamond N(Pw, z, t)\Diamond N(z, z, t)\Diamond N(z, z, t)\Diamond N(Pw, z, t),$

$N(Pw, z, kt) \leq N(Pw, z, t)$. Therefore by Lemma (2), we have $Pw = z$.

Since $STJUw = z$, thus we have $Pw = z = STJUw$, that is $w$ is a coincidence point of $P$ and $STJU$. This proves (v). Since $P(X) \subset ABIL(X), Pw = z$ implies that $z \in ABIL(X)$.

Let $v \in (ABIL)^{-1}z$. Then $ABILv = z$. By (iii), we have

$M(Py_n, Qv, kt) \geq M(ABILv, STJUy_n, t) * M(Py_n, STJUy_n, t) * M(Qv, ABILv, t) * M(Qv, STJUy_n, t) * M(Py_n, ABILv, t),$

$M(Py_n, Qv, kt) \geq M(z, STJUy_n, t) * M(Py_n, STJUy_n, t) * M(Qv, z, t) * M(Qv, STJUy_n, t) * M(Py_n, z, t)$.

Letting $n \to \infty$ , we have

$M(z, Qv, kt) \geq M(z, z, t) * M(z, z, t) * M(Qv, z, t)) * M(Qv, z, t) * M(z, z, t)$

and
$N(Py_n, Qv, kt) \leq N(ABILv, STJUy_n, t) \Diamond N(Py_n, STJUy_n, t) \Diamond N(Qv, ABILv, t) \Diamond N(Qv, STJUy_n, t) \Diamond N(Py_n, ABILv, t)$,

$N(Py_n, Qv, kt) \leq N(z, STJUy_n, t) \Diamond N(Py_n, STJUy_n, t) \Diamond N(Qv, z, t) \Diamond N(Qv, STJUy_n, t) \Diamond N(Py_n, z, t)$.

Letting $n \to \infty$ , we have

$N(z, Qv, kt) \leq N(z, z, t) \Diamond N(z, z, t) \Diamond N(Qv, z, t) \Diamond N(Qv, z, t) \Diamond N(z, z, t)$.

Therefore $M(z, Qv, kt) \geq M(Qv, z, t)$ and $N(z, Qv, kt) \leq N(Qv, z, t)$. Then by Lemma (2), we have $Qv = z$. Since $ABILv = z$, we have $Qv = z = ABILv$, i.e. $v$ is a coincidence point of $Q$ and $ABIL$. This proves (vi).

The remaining two cases pertain essentially to the previous cases. Indeed if $P(X)$ or $Q(X)$ is closed then by (i), $z \in P(X) \subset ABILX$ or $z \in Q(X) \subset STJU(X)$. Thus (v) and (vi) are completely established.

Since the pair $\{P, STJU\}$ is weakly compatible therefore $P$ and $STJU$ commute at their coincidence point that is $P(STJUw) = (STJU)Pw$ or $Pz = STJUz$. Consequently, the pair $\{Q, ABIL\}$ is weakly compatible, therefore $Q$ and $ABIL$ commute at their coincidence point that is $Q(ABILv) = (ABIL)Qv$ or $Qz = ABILz$. Now we prove that $Pz = z$. By (iii), we have

$M(Pz, Qx_n, kt) \geq M(ABILx_n, STJUz, t) * M(Pz, STJUz, t) * M(Qx_n, ABILx_n, t)$
$M(Qx_n, STJUz, t) * M(Pz, ABILx_n, t)$.

Proceeding to limit as $n \to \infty$ , we have
$M(Pz, z, kt) \geq M(z, Pz, t) * M(Pz, Pz, t) * M(z, z, t) * M(z, Pz, t) * M(Pz, z, t)$
and
$N(Pz, Qx_n, kt) \leq N(ABILx_n, STJUz, t) \Diamond N(Pz, STJUz, t) \Diamond N(Qx_n, ABILx_n, t) \Diamond N(Qx_n, STJUz, t) \Diamond N(Pz, ABILx_n, t)$.
Proceeding to limit as $n \to \infty$ , we have
$N(Pz, z, kt) \leq N(z, Pz, t) \Diamond N(Pz, Pz, t) \Diamond N(z, z, t) \Diamond N(z, Pz, t) \Diamond N(Pz, z, t)$.
Therefore, we have $M(Pz, z, kt) \geq M(z, Pz, t)$ and $N(Pz, z, kt) \leq N(z, Pz, t)$. By Lemma (2), we have $Pz = z$. So $Pz = STJUz = z$,
By (iii) we have
$M(Py_n, Qz, kt) \geq M(ABILz, STJUy_n, t) * M(Py_n, STJUy_n, t) * M(Qz, ABILz, t) * M(Qz, STJUy_n, t) * M(Py_n, ABILz, t)$.

Letting $n \to \infty$ , we have

$$M(z,Qz,kt) \geq M(z,z,t) * M(z,z,t) * M(Qz,z,t) * M(Qz,z,t) * M(z,z,t)$$

and
$N(Py_n,Qz,kt) \leq N(ABILz,STJUy_n,t) \Diamond N(Py_n,STJUy_n,t) \Diamond N(Qz,ABILz,t) \Diamond$
$N(Qz,STJUy_n,t) \Diamond N(Py_n,ABILz,t)$.
Letting $n \to \infty$ , we have

$$N(z,Qz,kt) \leq N(z,z,t) \Diamond N(z,z,t) \Diamond N(Qz,z,t) \Diamond N(Qz,z,t) \Diamond N(z,z,t).$$

Therefore, we have $M(z,Qz,kt) \geq M(Qz,z,t)$ and $N(z,Qz,kt) \leq N(Qz,z,t)$. Now by Lemma (2), we have $Qz = z$. So $Qz = ABILz = z$. By (iii) and using (vii), we have

$$M(Pz,Q(Lz),kt) \geq M(ABIL(Lz),STJUz,t) * M(Pz,STJUz,t) * M(Q(Lz),ABIL(Lz),t) *$$

$$M(Q(Lz),STJUz,t) * M(Pz,ABIL(Lz),t),$$

$$M(z,Lz,kt) \geq M(Lz,z,t) * M(z,z,t) * M(Lz,Lz,t) * M(Lz,z,t) * M(z,Lz,t)$$
and

$N(Pz,Q(Lz),kt) \leq N(ABIL(Lz),STJUz,t) \Diamond N(Pz,STJUz,t) \Diamond N(Q(Lz),ABIL(Lz),t) \Diamond$
$N(Q(Lz),STJUz,t) \Diamond N(Pz,ABIL(Lz),t)$.

$$N(z,Lz,kt) \leq N(Lz,z,t) \Diamond N(z,z,t) \Diamond N(Lz,Lz,t) \Diamond N(Lz,z,t) \Diamond N(z,Lz,t).$$

Therefore, we have $M(z,Lz,kt) \geq M(Lz,z,t)$ and $N(z,Lz,kt) \leq N(z,Lz,t)$. Now, by Lemma (2), we have $Lz = z$. Since $ABILz = z$, therefore $ABIz = z$. By (iii) and using (vii), we have

$M(Pz,Q(Iz),kt) \geq M(ABIL(Iz),STJUz,t) * M(Pz,STJUz,t) * M(Q(Iz),ABIL(Iz),t) *$
$M(Q(Iz),STJUz,t) * M(Pz,ABIL(Iz),t)$.

$$M(z,Iz,kt) \geq M(Iz,z,t) * M(z,z,t) * M(Iz,Iz,t) * M(Iz,z,t) * M(z,Iz,t)$$
and

$N(Pz,Q(Iz),kt) \leq N(ABIL(Iz),STJUz,t) \Diamond N(Pz,STJUz,t) \Diamond N(Q(Iz),ABIL(Iz),t)$
$\Diamond N(Q(Iz),STJUz,t) \Diamond N(Pz,ABIL(Iz),t)$.

$$N(z,Iz,kt) \leq N(Iz,z,t) \Diamond N(z,z,t) \Diamond N(Iz,Iz,t) \Diamond N(Iz,z,t) \Diamond N(z,Iz,t).$$

Thus we have $M(z,Iz,kt) \geq M(Iz,z,t)$ and $N(z,Iz,kt) \leq N(Iz,z,t)$. Now by Lemma (2), we have $Iz = z$. Since $ABIz = z$, therefore $ABz = z$. Now to prove $Bz = z$ we put $x = z, y = Bz$ in (iii) and using (vii), we have

 

$M(Pz, Q(Bz), kt) \geq M(ABIL(Bz), STJUz, t) * M(Pz, STJUz, t) * M(Q(Bz), ABIL(Bz), t) * M(Q(Bz), STJUz, t) * M(Pz, ABIL(Bz), t),$

$M(z, Bz, kt) \geq M(Bz, z, t) * M(z, z, t) * M(Bz, Bz, t) * M(Bz, z, t) * M(z, Bz, t)$
and

$N(Pz, Q(Bz), kt) \leq N(ABIL(Bz), STJUz, t) \Diamond N(Pz, STJUz, t) \Diamond N(Q(Bz), ABIL(Bz), t) \Diamond N(Q(Bz), STJUz, t) \Diamond N(Pz, ABIL(Bz), t).$

$N(z, Bz, kt) \leq N(Bz, z, t) \Diamond N(z, z, t) \Diamond N(Bz, Bz, t) \Diamond N(Bz, z, t) \Diamond N(z, Bz, t).$

Thus we have $M(z, Bz, kt) \geq M(Bz, z, t)$ and $N(z, Bz, kt) \leq N(Bz, z, t)$. Now by Lemma (2), we have $Bz = z$. Since $ABz = z$, therefore $Az = z$. By (iii) and using (vii), we have

$M(P(Uz), Qz, kt) \geq M(ABILz, STJU(Uz), t) * M(P(Uz), STJU(Uz), t) * M(Qz, ABILz, t) * M(Qz, STJU(Uz), t) * M(P(Uz), ABILz, t),$

$M(Uz, z, kt) \geq M(Uz, z, t) * 1 * 1 * M(Uz, z, t) * M(Uz, z, t \geq M(Uz, z, t)$
and

$N(P(Uz), Qz, kt) \leq N(ABILz, STJU(Uz), t) \Diamond N(P(Uz), STJU(Uz), t) \Diamond N(Q(Qz), ABIL(Qz), t) \Diamond N(Qz, STJU(Uz), t) \Diamond N(P(Uz), ABILz), t).$

$N(Uz, z, kt) \leq N(Uz, z, t) \Diamond 0 \Diamond 0 \Diamond N(Uz, z, t) \Diamond N(Uz, z, t) \leq N(Uz, z, t).$

Thus we have, $M(Uz, z, kt) \geq M(Uz, z, t)$ and $N(Uz, z, kt) \leq N(Uz, z, t)$. Therefore by Lemma (2), we have $Uz = z$. Since $STJUz = z$, therefore $STJz = z$. To Prove $Jz = z$ , put $x = Jz, y = z$ in (iii) and using (vii), we have

$M(P(Jz), Qz, kt) \geq M(ABILz, STJU(Jz), t) * M(P(Jz), STJU(Jz), t) * M(QZ, ABILz, t) * M(Qz, STJUz, t) * M(P(Jz), ABILz, t).$

$M(Jz, z, kt) \geq M(Jz, z, t) * 1 * 1 * M(Jz, z, t) * M(Jz, z, t \geq M(Jz, z, t)$
and

$N(P(Jz), Qz, kt) \leq N(ABILz, STJU(Jz), t) \Diamond N(P(Jz), STJU(Jz), t) \Diamond N(Qz, ABILz, t) \Diamond N(Qz, STJU(Jz), t) \Diamond N(P(Jz), ABILz, t).$

$N(Jz, z, kt) \leq N(Jz, z, t) \Diamond 0 \Diamond 0 \Diamond N(Jz, z, t) \Diamond N(Jz, z, t) \leq N(Jz, z, t).$

Thus we have, $M(Jz, z, kt) \geq M(Jz, z, t)$ and $N(Jz, z, kt) \leq N(Jz, z, t)$. Now by Lemma (2), we have $Jz = z$. Since $STJz = z$, therefore $STz = z$. To prove $Tz = z$ , put $x = Tz, y = z$ in (iii) and using (vii), we have

$M(P(Tz), Qz, kt) \geq M(ABILz, STJU(Tz), t) * M(P(Tz), STJU(Tz), t) *$
$M(Q(Tz), ABIL(Tz), t) * M(Q(Tz), STJU(Tz), t) * M(P(Tz), ABIL(Tz), t).$

$M(Tz, z, kt) \geq M(Tz, z, t) * 1 * 1 * M(Tz, z, t) * M(Tz, z, t \geq M(Tz, z, t)$
and

$N(P(Tz), Qz, kt) \leq N(ABILz, STJU(Tz), t) \Diamond N(P(Tz), STJU(Tz), t) \Diamond N(Qz, ABILz, t)$
$\Diamond N(Qz, STJU(Tz), t) \Diamond N(P(Tz), ABILz, t).$

$N(Tz, z, kt) \leq N(Tz, z, t) \Diamond 0 \Diamond 0 \Diamond N(Tz, z, t) \Diamond N(Tz, z, t) \leq N(Tz, z, t).$

Thus we have, $M(Tz, z, kt) \geq M(Tz, z, t)$ and $N(Tz, z, kt) \leq N(Tz, z, t)$. Now by Lemma (2), we have $Tz = z$. Since $STz = z$, therefore $Sz = z$. By combining the above results, we have

$Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z$ *i.e.* $z$ is a common fixed point of $A, B, S, T, I, J, L, U, P$ and $Q$.

For uniqueness of the common fixed point of $A, B, S, T, I, J, L, U, P$ and $Q$, let $u\,(u \neq z)$ be another common fixed point of $A, B, S, T, I, J, L, U, P$ and $Q$. By (iii), we have

$M(Pz, Qu, kt) \geq M(ABILu, STJUz, t) * M(Pz, STJUz, t) * M(Qu, ABILu, t) *$
$M(Qu, STJUz, t) * M(Pz, ABILu, t)$
and
$N(Pz, Qu, kt) \leq N(ABILu, STJUz, t) \Diamond N(Pz, STJUz, t) \Diamond N(Qu, ABILu, t)$
$\Diamond N(Qu, STJUz, t) \Diamond N(Pz, ABILu, t),$

$M(z, u, kt) \geq M(u, z, t) * M(z, z, t) * M(u, u, t) * M(u, z, t) * M(z, u, t)$
and

$N(z, u, kt) \leq N(u, z, t) \Diamond N(z, z, t) \Diamond N(u, u, t) \Diamond N(u, z, t) \Diamond N(z, u, t)$. Thus we have

$M(z, u, kt) \geq M(u, z, t)$ and $N(z, u, kt) \leq N(u, z, t)$. Therefore by lemma (2), we have $z = u$. This completes the proof of the theorem.
If $P = Q$ in Theorem 1. then we have the following.

**Corollary 1.** *Let $(X, M, N, *, \Diamond)$ be an intutionistic fuzzy metric space with continuous $t-$ norm $*$ and continuous t-conorm $\Diamond$ defined by $t * t \geq t$ and $(1-t) \Diamond (1-t) \leq (1-t), \forall t \in [0, 1]$ and let $A, B, S, T, I, J, L, U$ and $P$ be mappings from $X$ into itself such that*
(x) $P(X) \subset ABIL(X), P(X) \subset STJU(X)$,
(xi) $\{P, STJU\}$ *or* $\{P, ABIL\}$ satisfies the property $(S-B)$,
there exists a constant $k \in (0, 1)$ such that
(xii)$M(Px, Py, kt) \geq M(ABILy, STJUx, t) * M(Px, STJUx, t) * M(Py, ABILy, t)$
$* M(Py, STJUx, t) * M(Px, ABILy, t)$
and
$N(Px, Py, kt) \leq N(ABILy, STJUx, t) \Diamond N(Px, STJUx, t) \Diamond N(Py, ABILy, t) \Diamond N(Py, STJUx, t)$
$\Diamond N(Px, ABILy, t)$, $\forall x, y \in X$ and $t > 0$,

(xiii) if one of $P(X), ABIL(X), STJU(X)$ is a closed subspace of $X$, then
(xiv) $P$ and $STJU$ have a coincidence point and
(xv) $P$ and $ABIL$ have a coincidence point .
Further, if
(xvi) $AB = BA, AI = IA, AL = LA, BI = IB, BL = LB, IL = LI, PL = LP,$
$PI = IP, PB = BP, ST = TS, SJ = JS, SU = US, TJ = JT, TU = UT, JU = UJ, PU = UP, PJ = JP, PT = TP,$
(xvii) the pairs $\{P, STJU\}$ and $\{P, ABIL\}$ are weakly compatible, then
(xviii) $A, B, S, T, I, J, L, U$ and $P$ have a unique common fixed point in $X$.
If we put $L = U = I_x$ (The identity map on $X$) in Theorem 1., we have the following:

**Corollary 2.** *Let $(X, M, N, *, \Diamond)$ be an intutionistic fuzzy metric space with continuous $t-$ norm $*$ and continuous t-conorm $\Diamond$ defined by $t*t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t), \forall t \in [0,1]$ and let $A, B, S, T, I, J, P$ and $Q$ be mappings from $X$ into itself such that*
(xix) $P(X) \subset ABI(X), Q(X) \subset STJ(X)$,
(xx) $\{P, STJ\}$ *or* $\{Q, ABI\}$ satisfies the property $(S-B)$,
there exists a constant $k \in (0,1)$ such that
(xxi)$M(Px, Qy, kt) \geq M(ABIy, STJx, t) * M(Px, STJx, t) * M(Qy, ABIy, t) * M(Qy, STJx, t) * M(Px, ABIy, t)$
and
$N(Px, Qy, kt) \leq N(ABIy, STJx, t)\Diamond N(Px, STJx, t)\Diamond N(Qy, ABIy, t)\Diamond N(Qy, STJx, t) \Diamond N(Px, ABIy, t)$, $\forall x, y \in X$ and $t > 0$,
(xxii) if one of $P(X), ABI(X), STJ(X), Q(X)$ is a closed subspace of $X$, then
(xxiii) $P$ and $STJ$ have a coincidence point and
(xxiv) $Q$ and $ABI$ have a coincidence point .
Further, if
(xxv) $AB = BA, AI = IA, BI = IB, QI = IQ, QB = BQ, ST = TS, SJ = JS,$
$TJ = JT, PJ = JP, PT = TP,$
(xxvi) the pairs $\{P, STJ\}$ and $\{P, ABI\}$ are weakly compatible, then
(xxvii) $A, B, S, T, I, J, P$ and $Q$ have a unique common fixed point in $X$.
If we put $P = Q$ in Corollary (2.) ,we get the following

**Corollary 3.** *Let $(X, M, N, *, \Diamond)$ be an intutionistic fuzzy metric space with continuous $t-$ norm $*$ and continuous t-conorm $\Diamond$ defined by $t*t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t), \forall t \in [0,1]$ and let $A, B, S, T, I, J$ and $P$ be mappings from $X$ into itself such that*
(xxviii) $P(X) \subset ABI(X), P(X) \subset STJ(X)$,
(xxix) $\{P, STJ\}$ *or* $\{P, ABI\}$ satisfies the property $(S-B)$,
there exists a constant $k \in (0,1)$ such that
(xxx)$M(Px, Py, kt) \geq M(ABIy, STJx, t) * M(Px, STJx, t) * M(Py, ABIy, t) * M(Py, STJx, t) * M(Px, ABIy, t)$
and
$N(Px, Py, kt) \leq N(ABIy, STJx, t)\Diamond N(Px, STJx, t)\Diamond N(Py, ABIy, t)\Diamond N(Py, STJx, t) \Diamond N(Px, ABIy, t)$, $\forall x, y \in X$ and $t > 0$,
(xxxi) if one of $P(X), ABI(X), STJ(X)$, is a closed subspace of $X$, then
(xxxii) $P$ and $STJ$ have a coincidence point and
(xxxiii) $P$ and $ABI$ have a coincidence point .
Further, if

(xxxiv) $AB = BA, AI = IA, BI = IB, PI = IP, PB = BP, ST = TS, SJ = JS, TJ = JT, PJ = JP, PT = TP,$
(xxxv) the pairs $\{P, STJ\}$ and $\{P, ABI\}$ are weakly compatible, then
(xxxvi) $A, B, S, T, I, J$ and $P$ have a unique common fixed point in $X$.

If we put $J = I = I_x$ (The identity map in $X$) in Corollary (3.) we have the following

**Corollary 4.** *Let $(X, M, N, *, \Diamond)$ be an intutionistic fuzzy metric space with continuous $t-$ norm $*$ and continuous t-conorm $\Diamond$ defined by $t*t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t), \forall t \in [0, 1]$ and let $A, B, S, T$ and $P$ be mappings from $X$ into itself such that*
(xxxvii) $P(X) \subset AB(X), P(X) \subset ST(X),$
(xxxviii) $\{P, ST\}$ *or* $\{P, AB\}$ satisfies the property $(S - B)$,
there exists a constant $k \in (0, 1)$ such that
(xxxix)$M(Px, Py, kt) \geq M(ABy, STx, t)*M(Px, STx, t)*M(Py, ABy, t)*M(Py, STx, t)* M(Px, ABy, t)$
and
$N(Px, Py, kt) \leq N(ABy, STx, t)\Diamond N(Px, STx, t)\Diamond N(Py, ABy, t)\Diamond N(Py, STx, t)\Diamond N(Px, ABy, t),$
$\forall x, y \in X$ and $t > 0$,
(xxxx) if one of $P(X), AB(X), ST(X)$, is a closed subspace of $X$, then
(xxxxi) $P$ and $ST$ have a coincidence point and
(xxxxii) $P$ and $AB$ have a coincidence point .
Further, if
(xxxxiv) $AB = BA, PB = BP, ST = TS, PT = TP,$
(xxxxv) the pairs $\{P, ST\}$ and $\{P, AB\}$ are weakly compatible, then
(xxxxvi) $A, B, S, T$ and $P$ have a unique common fixed point in $X$.

**Conflict of Interests**

*The author declare that there is no conflict of interests regarding the publication of this paper.*

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