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GLOBAL STABILITY OF HIGHLY PATHOGENIC AVIAN INFLUENZA EPIDEMIC MODEL WITH VERTICAL TRANSMISSION FUNCTION

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Abstract In this paper, we introduce a basic reproduction number for a highly pathogenic avian influenza epidemic model with vertical transmission function and emigration rate. Then we establish that global dynamics are completely determined by the basic reproduction number σ. We show that; if σ is less than or equal to one there is only one disease free equilibrium which is globally stable and disease dies out, whereas if σ is greater than one, there is a unique endemic equilibrium which is globally stable and thus the disease persists. Finally a numerical example is also included to illustrate the effectiveness of the proposed model.

MSC: 34D23, 93A30, 93D20

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1. Introduction

In 1918, the great pandemic of influenza in modern history occurred. About 40000000 humans died of this pandemic all over the world. After 1918, a pandemic of influenza occurred twice in 1957 and 1968. The highly pathogenic Avian Influenza has a high death rate, which is about 100 percent for birds and more than 70 percent for humans analyzed by Iwami [7]. We found that Avian Influenza virus subtypes which can directly infect human are: H5N1, H7N1, H7N2, H7N3, H7N7, H9N2, H7N9 subtype. Among them, the new subtype Avian influenza virus H7N9 subtype, which was first discovered in 2013 march, and the high pathogenic Avian Influenza H5N1 subtype are particular noteworthy. The Avian influenza virus not only caused human causalities but also hit the poultry industry.

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In Hong Kong in 1997, the news that a human is infected with avian influenza from birds was reported. After that, infection to human of avian influenza occurred successively. It is known that already 133 humans have been infected in Asia since late 2003 and it killed 68. In humans, avian influenza virus causes the similar symptoms as the other types of influenza. These include fever, cough, sore throat, muscle aches, conjunctivitis and, in severe cases, severe breathing problems and pneumonia that may be fatal.

By March 2013, the world has reported a total of human infection of highly pathogenic H5N1 Avian influenza in 622 cases, including 371 deaths. The distribution of cases in 15 countries, including China is found in 45 cases, 30 cases of death. Most of human being infected with H5N1 Avian influenza are young people and children. In March 2013, human infection with H7N9 Avian influenza was first found in China. By May 1, 2013 Shanghai, Anhai, Jiangsu, Zhejing, Beijing, Henan, Shandong, Jiangxi, Hunan, Fujian and other 10 city have reported 127 confirmed case, including 26 death cases studied by Che [1]. The number of mathematical modeling studies have been carried out to quantify the potential burden of an influenza pandemic in human being and to assess various control strategies considered by et al. [2, 3, 4, 6, 8, 9, 10, 11, 14]. Avian influenza modeling studies involving humans and birds was carried out in Gumel [5] and Iwami [7]. Shivram et al. [12, 13] proposed an SIR model with emigration rate and disease related death rate. In this paper we consider highly pathogenic avian influenza epidemic model with emigration rate and vertical transmission function. In the next section, we present the model. In third section we derive the disease free equilibrium and the endemic equilibrium. In the fourth section, we prove some theorems for the global stability of the disease free and endemic equilibrium. The last section contains a numerical simulation and discussion.

2. Mathematical Model

In this paper, we propose an avian influenza model with emigration rate and vertical transmission function. Our proposed model is as follows:

$$\begin{aligned}
\frac{dX}{dt} &= a_1 - \frac{\omega XY}{1+\delta Y} - d_1 X - b_1 \\
\frac{dY}{dt} &= \frac{\omega XY}{1+\delta Y} - (d_1+\alpha_1)Y \\
\frac{dS}{dt} &= a_2 - \frac{\beta SY}{1+\delta Y} - d_2 S - pa_2 I - b_2 \\
\frac{dI}{dt} &= \frac{\beta SY}{1+\delta Y} - (\alpha_2+d_2+\gamma)I + pa_2 I \\
\frac{dR}{dt} &= \gamma I - d_2 R
\end{aligned} \tag{2.1}$$

In system (1.1), the human is divided into three compartments; susceptible (S), infected (I), recovered (R). The birds are divided into susceptible poultry (X) and infected poultry (Y). The parameters a_1, a_2 are respectively the natural birth rate of Avian and human such that $a_1 > b_1$ and $a_2 > b_2$. d_1 and d_2 are respectively the natural mortality of poultry and human. α_1 and α_2 are respectively the poultry and human mortality due to illness. ω is a infectious rate of susceptible poultry to infected poultry. β is a infected poultry of the infection rate of susceptible individuals, γ is the recovery rate that infects individuals through treatments, δ is a parameter which is effect of infectious disease when the contact rate of disease is saturated, p is a constant. The transmission process of the disease is shown in the figure.

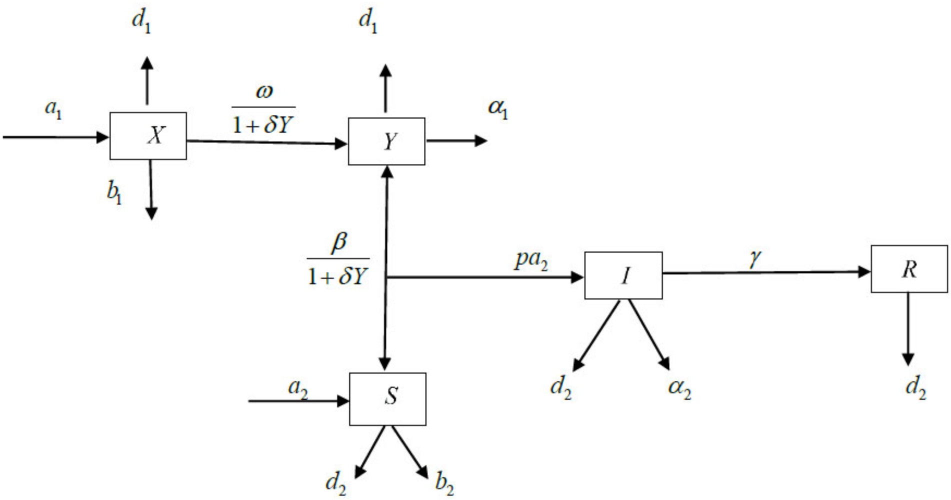


FIGURE 1. Flow diagram

Where $a_1 > b_1$ and $a_2 > b_2$

3. Equilibrium analysis

It can be checked that the system (1.1) has two non-negative equilibrium and one of the disease free equilibrium $E_0\left(\frac{a_1-b_1}{d_1}, 0, \frac{a_2-b_2}{d_2}, 0\right)$.

Existence of $E_+\left(X^*, Y^*, S^*, I^*\right)$
Here X^*, Y^*, S^*, I^* are the positive solution of the following algebraic equation,

$$\begin{aligned} a_1 - b_1 - \frac{\omega XY}{1+\delta Y} - d_1X &= 0 \\ \frac{\omega X}{1+\delta Y} - (d_1+\alpha_1) &= 0 \\ a_2 - b_2 - \frac{\beta SY}{1+\delta Y} - d_2S - pa_2I &= 0 \\ \frac{\beta SY}{1+\delta Y} - (\alpha_2+d_2+\gamma)I + pa_2I &= 0 \end{aligned} \tag{3.1}$$

Solving (2.1) we get

$$X^* = \frac{d_1+\alpha_1+(a_1-b_1)\delta}{d_1\delta+\omega}, \; Y^* = \frac{(a_1-b_1)\omega - d_1(d_1+\alpha_1)}{(\omega+d_1\delta)(d_1+\alpha_1)}$$

$$S^* = \frac{(a_2-b_2)(1+\delta Y^*)(\alpha_2+d_2+\gamma-pa_2)}{[\beta Y^* + d_2(1+\delta Y^*)][\alpha_2+d_2+\gamma-pa_2] + pa_2\beta Y^*}$$

$$I^* = \frac{\beta(a_2-b_2)Y^*}{[\beta Y^* + d_2(1+\delta Y^*)][\alpha_2+d_2+\gamma-pa_2] + pa_2\beta Y^*}$$

Where $pa_2 < (\alpha_2 + d_2 + \gamma)$.
We can get the basic reproductive number of the system (2.1)

$$\sigma = \frac{(a_1 - b_1)\omega}{d_1(d_1 + \alpha_1)}$$

Hence if $\sigma \leq 1$, the system (2.1) only exists the disease-free equilibrium $E_0\left(\frac{a_1-b_1}{d_1}, 0, \frac{a_2-b_2}{d_2}, 0\right)$, when $\sigma > 1$ there exists only one endemic equilibrium

$$E_+\left(\begin{array}{c} X^* = \frac{d_1+\alpha_1+(a_1-b_1)\delta}{d_1\delta+\omega}, Y^*, S^* = \frac{(a_2-b_2)(1+\delta Y^*)(\alpha_2+d_2+\gamma-pa_2)}{[\beta Y^*+d_2(1+\delta Y^*)][\alpha_2+d_2+\gamma-pa_2]+pa_2\beta Y^*} \\ \\ I^* = \frac{\beta(a_2-b_2)Y^*}{[\beta Y^*+d_2(1+\delta Y^*)][\alpha_2+d_2+\gamma-pa_2]+pa_2\beta Y^*} \end{array}\right)$$

Where

$$Y^* = \frac{(a_1 - b_1)\omega - d_1(d_1 + \alpha_1)}{(\omega + d_1\delta)(d_1 + \alpha_1)}.$$

4. Linear Stability Analysis

Theorem 4.1. *The disease free equilibrium J_{E_0} is locally asymptotically stable if $\sigma \leq 1$, and disease free equilibrium E_0 is unstable if $\sigma > 1$.*

Proof. The Jacobian matrix of system (2.1) is

$$J = \begin{bmatrix} -d_1 - \frac{\omega Y}{1+\delta Y} & -\frac{\omega X(1+\delta Y)-\delta\omega XY}{(1+\delta Y)^2} & 0 & 0 \\ \frac{\omega Y}{1+\delta Y} & \frac{\omega X(1+\delta Y)-\delta\omega XY}{(1+\delta Y)^2} - (d_1 + \alpha_1) & 0 & 0 \\ 0 & \frac{-\beta S(1+\delta Y)-\delta\beta SY}{(1+\delta Y)^2} & \frac{-\beta Y}{1+\delta Y} - d_2 & -pa_2 \\ 0 & \frac{\beta S(1+\delta Y)-\delta\beta SY}{(1+\delta Y)^2} & \frac{\beta Y}{1+\delta Y} & -(d_2 + \alpha_2 + \gamma) + pa_2 \end{bmatrix}$$

Now, Jacobian matrix of system (2.1) at $E_0\left(\frac{a_1-b_1}{d_1}, 0, \frac{a_2-b_2}{d_2}, 0\right)$, is

$$J_{E_0} = \begin{bmatrix} -d_1 & -\frac{\omega(a_1-b_1)}{d_1} & 0 & 0 \\ 0 & \frac{\omega(a_1-b_1)}{d_1} - (d_1 + \alpha_1) & 0 & 0 \\ 0 & \frac{-\beta(a_2-b_2)}{d_2} & -d_2 & -pa_2 \\ 0 & \frac{\beta(a_2-b_2)}{d_2} & 0 & -(\alpha_2 + d_2 + \gamma) + pa_2 \end{bmatrix}$$

The characteristic equation is

$$\begin{vmatrix} -d_1 - \lambda & -\frac{\omega(a_1-b_1)}{d_1} & 0 & 0 \\ 0 & \frac{\omega(a_1-b_1)}{d_1} - (d_1 + \alpha_1) - \lambda & 0 & 0 \\ 0 & \frac{-\beta(a_2-b_2)}{d_2} & -d_2 - \lambda & -pa_2 \\ 0 & \frac{\beta(a_2-b_2)}{d_2} & 0 & -(\alpha_2 + d_2 + \gamma) + pa_2 - \lambda \end{vmatrix} = 0$$

$$(d_1 + \lambda)(d_2 + \lambda)(\alpha_2 + d_2 + \gamma - pa_2 + \lambda)\left[\left\{\frac{(a_1 - b_1)\omega}{\alpha_1} - (d_1 + \alpha_1)\right\} - \lambda\right] = 0 \quad (4.1)$$

The roots of (4.1) are

$$-d_1, -d_2, \quad -(\alpha_2 + d_2 + \gamma) + pa_2, \quad \frac{\omega(a_1 - b_1)}{d_1} - (d_1 + \alpha_1)$$

The first three roots having negative real parts and fourth root $\frac{\omega(a_1-b_1)}{d_1} - (d_1 + \alpha_1)$ will have negative real part if $\sigma \leq 1$. Thus all roots of (4.1) have negative real parts so E_0 is locally asymptotically stable if $\sigma \leq 1$, and the root $\frac{\omega(a_1-b_1)}{d_1} - (d_1 + \alpha_1)$ will have positive real part if $\sigma > 1$, so E_0 is an unstable.

Theorem 4.2. *The disease free equilibrium E_0 is globally asymptotically stable if $\sigma \leq 1$.*

Proof. Consider the Lyapunov function

$$\begin{aligned}
L_1 &= X - X^0 \ \ln X + Y \\
&= X^{'} - \frac{X^0}{X} X^{'} + Y^{'} \\
&= \Big(1 - \frac{X^0}{X}\Big) X^{'} + Y^{'} \\
&= \Big(1 - \frac{X^0}{X}\Big)\Big[a_1 - b_1 - \frac{\omega XY}{1+\delta Y} - d_1 X\Big] + \frac{\omega XY}{1+\delta Y} - (d_1 + \alpha_1)Y \\
&= \Big(1 - \frac{X^0}{X}\Big)\Big[d_1 X^0 - d_1 X - \frac{\omega XY}{1+\delta Y}\Big] + \frac{\omega XY}{1+\delta Y} - (d_1 + \alpha_1)Y \\
&\leq \frac{-d_1(X - X^0)^2}{X} + (d_1 + \alpha_1)Y\Big(\frac{\omega X^0}{(d_1 + \alpha_1)} - 1\Big) \\
&= \frac{-d_1(X - X^0)^2}{X} + (d_1 + \alpha_1)Y(\sigma - 1).
\end{aligned}$$

When $\sigma \leq 1$, we can get $L_1^{'} \leq 0$ and $L_1^{'} = 0$ has no other closed trajectory in addition to E_0 is globally asymptotically stable if and only if $\sigma \leq 1$

Theorem 4.3. *The endemic equilibrium E_+ is locally asymptotically stable if $\sigma > 1$.*

Proof. The Jacobian matrix of system (2.1) at $E_+(X^*, Y^*, S^*, I^*)$ is

$$J_{E_+} = \begin{bmatrix}
-d_1 - \frac{\omega Y^*}{1+\delta Y^*} & -\frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} & 0 & 0 \\
\frac{\omega Y^*}{1+\delta Y^*} & \frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} - (d_1+\alpha_1) & 0 & 0 \\
0 & \frac{-\beta S^*(1+\delta Y^*) - \delta\beta S^* Y^*}{(1+\delta Y^*)^2} & \frac{-\beta Y^*}{1+\delta Y^*} - d_2 & -pa_2 \\
0 & \frac{\beta S^*(1+\delta Y^*) - \delta\beta S^* Y^*}{(1+\delta Y^*)^2} & \frac{\beta Y^*}{1+\delta Y^*} & -(d_2+\alpha_2+\gamma)+pa_2
\end{bmatrix}$$

$$J_{E_+} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

Where

$$A = \begin{bmatrix}
-d_1 - \frac{\omega Y^*}{1+\delta Y^*} & -\frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} \\
\frac{\omega Y^*}{1+\delta Y^*} & \frac{\omega X^*(1+\delta Y^*) - \delta\omega X^* Y^*}{(1+\delta Y^*)^2} - (d_1 + \alpha_1)
\end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -\frac{\beta S^*(1+\delta Y^*)-\delta\beta S^* Y^*}{(1+\delta Y^*)^2} \\ 0 & \frac{\beta S^*(1+\delta Y^*)-\delta\beta S^* Y^*}{(1+\delta Y^*)^2} \end{bmatrix}, C = \begin{bmatrix} \frac{-\beta Y^*}{1+\delta Y^*} - d_2 & -pa_2 \\ \frac{\beta Y^*}{1+\delta Y^*} & -(d_2+\alpha_2+\gamma)+pa_2 \end{bmatrix}$$

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus J_{E_+} evaluated is stable if and only if so are A and C. The characteristic equations of the matrix A is

$$\lambda^2 + h_1\lambda + h_2 = 0,$$

Where $h_1 = d_1 + (d_1+\alpha_1)\frac{\delta Y^*}{1+\delta Y^*} + \frac{\omega Y^*}{1+\delta Y^*}$, $h_2 = (\omega + d_1\delta)(d_1+\alpha_1)\frac{Y^*}{1+\delta Y^*}$
Here $1+\delta Y^* > 0$ when $\sigma > 1$ so when $\sigma > 1, h_1, h_2 > 0$ by the Hurwitz criterion.
The characteristic roots of matrix A have negative real parts.
The characteristic equation of the matrix C

$$\left(\lambda + \frac{\beta Y^*}{1+\delta Y^*} + d_2\right)\left(\lambda + \alpha_2 + d_2 + \gamma - pa_2\right) = 0.$$

The characteristic roots of C are, $-\frac{\beta Y^*}{1+\delta Y^*} - d_2,\ -\left(\alpha_2 + d_2 + \gamma - pa_2\right)$
When $\sigma > 1$, The characteristic roots of C have negative real parts.
So all characteristic roots of the Jacobian matrix J_{E_+} have negative real parts if and only if $\sigma > 1$. Thus the endemic equilibrium E_+ is locally asymptotically stable if $\sigma > 1$.

Theorem 4.4. *The endemic equilibrium E_+ is globally asymptotically stable if $\sigma > 1$.*

Proof. Consider the Lyapunov function

$$L_2 = X^*\left(\frac{X}{X^*} - 1 - \ln\frac{X}{X^*}\right) + Y^*\left(\frac{Y}{Y^*} - 1 - \ln\frac{Y}{Y^*}\right)$$

Then $$L_2^{'} = X^*\left(\frac{X^{'}}{X^*} - \frac{X^*}{X}\cdot\frac{X^{'}}{X^*}\right) + Y^*\left(\frac{Y^{'}}{Y^*} - \frac{Y^*}{Y}\frac{Y^{'}}{Y^*}\right)$$

$$L_2^{'} = \left(1 - \frac{X^*}{X}\right)X^{'} + \left(1 - \frac{Y^*}{Y}\right)Y^{'} = C\left(2 - \frac{X^*}{X} - \frac{X}{X^*}\right).$$

By the relationship of arithmetic mean and geometric mean.
We know that

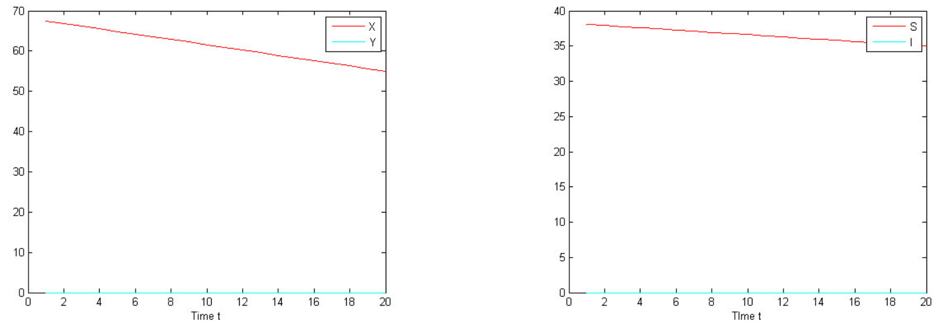
$$2 - \frac{X^*}{X} - \frac{X}{X^*} \leq 0.$$

i.e. $L_2^{'} \leq 0$, if and only if $(X,Y) = (X^*,Y^*), L_2^{'} = 0$. Thus by LaSalle invariance principle $E_+(X^*,Y^*,S^*,I^*)$ is globally asymptotically stable.

5. Simulation and discussion

In this paper we have discussed the global stability of highly pathogenic avian influenza epidemic model with vertical transmission function. Vertical transmission function is taken to represent the interaction between susceptible and infected poultry. To illustrate the results numerically, choose $a_1 = 3, \beta = 0.02, d_1 = 0.04, \alpha_1 = 0.96, a_2 = 2, d_2 = 0.052, \alpha_2 = 0.64, \gamma = 0.302, \delta = 0.5$, the emigration rate b_1, in poultry increase from 0.3

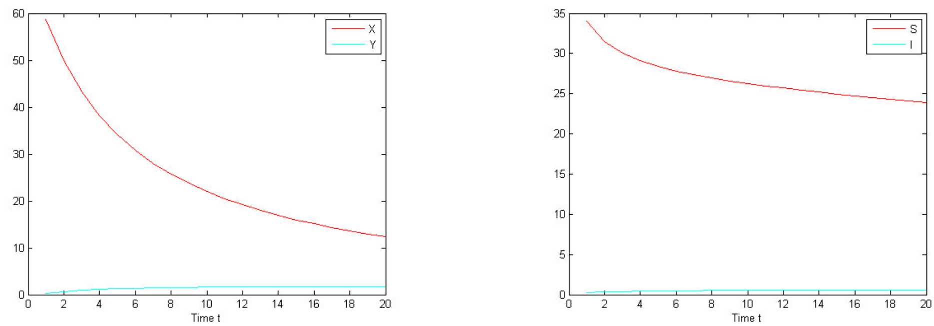
to 0.8, the emigration rate b_2 in human being increases from 0.02 to 0.18, parameter p varies from 0.02 to 0.12 and ω increases from 0.002 to 0.015. The basic reproductive number σ increases to 0.825 which is always less than 1.



(a) (b)

FIGURE 2.

Thus figure 2, (a) and (b) shows that the disease free equilibrium E_0 is globally asymptotically stable. Now we take the parameter of the system $a_1 = 3, \beta = 0.02, d_1 = 0.04, \alpha_1 = 0.96, a_2 = 2, d_2 = 0.052, \alpha_2 = 0.64, \gamma = 0.302, \delta = 0.5$, the emigration rate b_1 in poultry increases from 0.3 to 0.8, the emigration rate b_2 in human being, increases from 0.02 to 0.18, parameter p varies from 0.02 to 0.12 and ω increases from 0.02 to 0.15. The basic reproductive number σ increases to 8.25 which is always greater than 1.



(a) (b)

FIGURE 3.

Thus figure 3, (a) and (b) shows that all four compartments, $X(t), Y(t), S(t)$ and $I(t)$ approach to their steady state values, the disease becomes endemic.

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