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Riesz Triple Almost Lacunary Cesàro C_{111} statistical convergence of χ^3 defined by a Orlicz function

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Abstract In this paper we introduce a new concept for Riesz almost lacunary Cesàro statistical convergence of χ^3 sequence spaces strong $P-$ convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of Riesz almost lacunary Cesàroof χ^3 sequence spaces and also some inclusion theorems are discussed.

MSC: 40F05, 40J05, 40G05.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in *Apostol [1]* and double sequence spaces is found in *Hardy [7]*, *Subramanian et al. [8]*, *Deepmala et al. [9,10,16]* and many others. Later on investigated by some initial work on triple sequence spaces is found in *sahinar et al. [11]*, *Esi et al. [2-5]*, *Savas et al. [6]* , *Subramanian et al. [12]*, *Prakash et al. [13-14]*, Mishra et al. [17] and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$

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is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence $(S_{mnk})$is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m,n,k = 1,2,3,...) .$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m,n,k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y) = sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,... \right\}, \tag{1.1}$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ $in\,\Gamma^3$. Let $\phi = \{finite\ sequences\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m,n,k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\Im_{ijq}$ for all $m,n,k \in \mathbb{N}$,

$$\Im_{ijq} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{bmatrix}$$

with 1 in the $(i,j,q)^{th}$ position and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple gai sequence if $((m+n+k)!\,|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0$ as $m,n,k \to \infty$. The triple gai sequences will be denoted by χ^3.

2. Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P - limx = 0$) (i.e) $((m+n+k)!\,|x_{mnk}|)^{1/m+n+k} \to 0$ as $m,n,k \to \infty$. We shall write more briefly as $P - convergent\ \ to\ \ 0$.

2.1. Definition

A modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0$, $y \ge 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

2.2. Definition

A triple sequence $x = (x_{mnk})$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$P - lim_{p,q,u\to\infty} sup_{r,s,t\geq 0} \frac{1}{pqu} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} \sum_{k=t}^{t+u-1} \left((m+n+k)! \left|x_{mnk}\right|\right)^{1/m+n+k} \rightarrow 0.$$

that is, the average value of (x_{mnk}) taken over any rectangle $\{(m,n,k) : r \leq m \leq r+p-1, s \leq n \leq s+q-1, t \leq k \leq t+u-1\}$ tends to 0 as both p, q and u to ∞, and this $P-$ convergence is uniform in i, ℓ and j. Let denote the set of sequences with this property as $\left[\widehat{\chi^3}\right]$.

2.3. Definition

Let $(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$ be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0\ldots \\ q_{21} & q_{22} & \cdots & q_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = q_{11} + q_{12} + \ldots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \neq 0.$$ Then the transformation is given by

$T_{rst} = \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left((m+n+k)! \left|x_{mnk}\right|\right)^{1/m+n+k}$ is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - lim x = 0$.

2.4. Definition

The triple sequence $\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by
$I_{i,\ell,j} = \{(m,n,k) : m_{i-1} < m < m_i \, and \, n_{\ell-1} < n \leq n_\ell \, and \, k_{j-1} < k \leq k_j\}, q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}$.
Using the notations of lacunary sequence and Riesz mean for triple sequences.
$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that $Q_{m_i} = \sum_{m\in(0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k\in(0,k_j]} p_{k_j}$ and $H_i = \sum_{m\in(0,m_i]} p_{m_i}, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j}$. Clearly, $H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}$. If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell} \to \infty$ as $\ell \to \infty, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j} \to \infty$ as $j \to \infty$, then $\theta'_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \{(Q_{m_i} Q_{n_j} Q_{k_k})\}$ is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty, \overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences $(m_i), (n_\ell)$ and (k_j) are integers.
Throughout the paper, we assume that $Q_r = q_{11} + q_{12} + \ldots + q_{rs} \to \infty \, (r \to \infty), \overline{Q}_s = \overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \to \infty \, (s \to \infty), \overline{\overline{Q}}_t = \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \ldots + \overline{\overline{q}}_{rs} \to \infty \, (t \to \infty)$, such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as $i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$ as $j \to \infty$.
Let $Q_{m_i,n_\ell,k_j} = Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}, H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j$,
$I'_{i\ell j} = \left\{(m,n,k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \; and \; \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j}\right\}$,
$V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}}$ and $\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$. and $V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j$.
If we take $q_m = 1, \overline{q}_n = 1 \; and \; \overline{\overline{q}}_k = 1$ for all m, n and k then $H_{i\ell j}, Q_{i\ell j}, V_{i\ell j}$ and $I'_{i\ell j}$ reduce to $h_{i\ell j}, q_{i\ell j}, v_{i\ell j}$ and $I_{i\ell j}$.

Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:
$[\chi^3_R, \theta_{i\ell j}, q, f, p] =$
$\left\{P - lim_{i,\ell,j\to\infty} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell\in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0\right\}$,
uniformly in i, ℓ and j.
$[\Lambda^3_R, \theta_{i\ell j}, q, f, p] =$
$\left\{x = (x_{mnk}) : P - sup_{i,\ell,j} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell\in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left|x_{m+i,n+\ell,k+j}\right|^{p_{mnk}}\right] < \infty\right\}$,
uniformly in i, ℓ and j.
Let f be an Orlicz function, $p = p_{mnk}$ be any factorable triple sequence of strictly positive real numbers and and $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$,
If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k, then we obtain the following sequence spaces.
$[\chi^3_R, q, f, p] =$
$\left\{P - lim_{r,s,t\to\infty} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0\right\}$,
uniformly in i, ℓ and j.
$[\Lambda^3_R, q, f, p] =$

$\left\{P - sup_{r,s,t}\frac{1}{Q_r\overline{Q}_s\overline{\overline{Q}}_t}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] < \infty\right\}$, uniformly in i, ℓ and j.

2.5. Definition

Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space:
$\theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\}$ be a triple lacunary sequence $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] =$
$\left\{P - lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0,\right\}$,
uniformly in i, ℓ and j.

We shall denote $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right]$ as $\chi^3\left[AC_{\theta_{i,\ell,j}}, p\right]$ respectively when $p_{mnk} = 1$ for all m, n and k If x is in $\chi^3\left[AC_{\theta_{i,\ell,j}}, p\right]$, we shall say that x is almost lacunary χ^3 strongly $p-$convergent with respect to the Orlicz function f. Also note if $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] = \chi^3\left[AC_{\theta_{i,\ell,j}}\right]$ which are defined as follows:
$\chi^3\left[AC_{\theta_{i,\ell,j}}\right] =$
$\left\{P - lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right] = 0,\right\}$,
uniformly in i, ℓ and j.
Again note if $p_{mnk} = 1$ for all m, n and k then $\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] = \chi^3_f\left[AC_{\theta_{i,\ell,j}}\right]$. we define
$\chi^3_f\left[AC_{\theta_{i,\ell,j}}, p\right] =$
$\left\{P - lim_{i,\ell,j}\frac{1}{h_{i\ell j}}\sum_{m\in I_{k,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}}\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0,\right\}$,
uniformly in i, ℓ and j.

2.6. Definition

Let f be an Orlicz function $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence space: $\chi^3_f[p] =$
$\left\{P - lim_{r,s,t\to\infty}\frac{1}{rst}\sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{k=1}^{t}\left[f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0\right\}$,
uniformly in i, ℓ and j.

If we take $f(x) = x, p_{mnk} = 1$ for all m, n and k then $\chi^3_f[p] = \chi^3$.

2.7. Definition

Let $\theta_{i,\ell,j}$ be a triple lacunary sequence; the triple sequence x is $\widehat{S_{\theta i,\ell,j}} - p-$ convergent to 0 then
$P - lim_{i,\ell,j}\frac{1}{h_{i,\ell,j}}max_{i,\ell,j}\left|\left\{(m,n,k)\in I_{i,\ell,j} : f\left((m+n+k)!\left|x_{m+i,n+\ell,k+j} - 0\right|\right)^{1/m+n+k}\right\}\right| =$
0.
In this case we write $\widehat{S_{\theta i,\ell,j}} - lim\left(f\left(m+n+k\right)!\left|x_{m+i,n+\ell,k+j} - 0\right|\right)^{1/m+n+k} = 0$.

3. Lacunary CeÀro $C_{111}-$ statistical convergence of triple sequences

Let $A = [a^{pqr}_{mnk}]^{\infty}_{m,n,k=0}$ be a triple infinite matrix of real number for $p, q, r = 1, 2, \cdots$ forming the sum

$$y_{pqr} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a^{pqr}_{mnk}x_{mnk} \tag{3.1}$$

Called the A means of the triple sequence x yielded a method of summability. We say that a sequence x is A summable to the limit 0 of the A mean exist for all $p, q, r = 0, 1, \cdots$ and converges.

$$lim_{uvw \to \infty} \sum_m^u \sum_n^v \sum_k^w a_{mnk}^{pqr} x_{mnk} = y_{pqr}$$

and

$$\lim_{pqr \to \infty} y_{pqr} = 0$$

Define the means

$$\sigma_{pqr}^x = \frac{1}{pqr} \sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r x_{mnk}$$

and

$$A\sigma_{pqr}^x = \frac{1}{pqr} \sum_{m=0}^p \sum_{n=0}^q \sum_{k=0}^r a_{mnk}^{pqr} x_{mnk}.$$

We say that $x = (x_{mnk})$ is statistical summable $(C, 1, 1, 1)$ to 0, if the sequence $\sigma = (\sigma_{mnk}^x)$ is statistically convergent to 0, that is, $st_3 - lim_{pqr} \sigma_{pqr}^x = 0$. It is denoted by $C_{111}(st_3)$, the set of all triple sequence which one statistically summable $(C, 1, 1, 1)$.

4. Main Results

4.1. Theorem

If f be any Orlicz function and a bounded factorable positive triple number sequence p_{mnk} then $\sigma^{\chi_f^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}}, P \right]$ is linear space

Proof: The proof is easy. Theorefore omit the proof.

4.2. Theorem

For any Orlicz function f, we have

$\sigma^{\chi^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}} \right] \subset \sigma^{\chi_f^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}} \right]$

Proof: Let $x \in \sigma^{\chi^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}} \right]$ so that for each i, ℓ and j

$\sigma^{\chi^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}} \right] = lim_{i,\ell,j} \frac{1}{pqr} \frac{1}{h_{i\ell j}} \sum_{m=0, m \in I_{i,\ell,j}}^p \sum_{n=0, n \in I_{i,\ell,j}}^q \sum_{k=0, k \in I_{i,\ell,j}}^r$

$a_{mnk}^{pqr} \left[\left((m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right)^{1/m+n+k} \right] = 0.$

Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \leq t \leq \delta$. We obtain the following,

$\frac{1}{h_{i\ell j}} (h_{i\ell j} \epsilon) + \frac{1}{pqr, h_{i\ell j}} \sum_{m=0, m \in I_{i,\ell,j}}^p \sum_{n=0, n \in I_{i,\ell,j}}^q \sum_{k=0, k \in I_{i,\ell,j} \, and \, |x_{m+i, n+\ell, k+j} - 0| > \delta}^r$

$a_{mnk}^{pqr} f \left[\left((m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right)^{1/m+n+k} \right]$

$\frac{1}{h_{i\ell j}} (h_{i\ell j} \epsilon) + \frac{pqr}{h_{i\ell j}} K \delta^{-1} a_{mnk}^{pqr} f(2) h_{i\ell j} \, \sigma^{\chi^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}} \right].$

Hence i, ℓ and j goes to infinity, we are granted $x \in \sigma^{\chi_f^3} \left[A\sigma_{pqr} C_{\theta_{i,\ell,j}} \right].$

4.3. Theorem

Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence with $liminf_i q_i > 1, \ liminf_\ell \overline{q_\ell} > 1 \ and \ liminf_j q_j > 1$ then for any Orlicz function f, $\sigma^{\chi_f^3}(P) \subset \sigma^{\chi_f^3}\left(A\sigma_{pqr} C_{\theta_{i,\ell,j}}, P \right)$

Proof: Suppose $liminf_i q_i > 1, \ liminf_\ell \overline{q_\ell} > 1$ and $liminf_j q_j > 1$ then there exists $\delta > 0$ such that $q_i > 1 + \delta, \ \overline{q_\ell} > 1 + \delta$ and $q_j > 1 + \delta$ This implies $\frac{h_i}{m_i} \geq \frac{\delta}{1+\delta}, \ \frac{h_\ell}{n_\ell} \geq \frac{\delta}{1+\delta}$ and $\frac{h_j}{k_j} \geq \frac{\delta}{1+\delta}$ Then for $x \in \sigma^{\chi_f^3}(P)$, we can write for each r, s and u.

$B_{i,\ell,j} = \frac{1}{pqr, h_{i\ell j}} \sum_{m=0, m \in I_{i,\ell,j}}^p \sum_{n=0, n \in I_{i,\ell,j}}^q \sum_{k=0, k \in I_{i,\ell,j}}^r$

$a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} =$
$\frac{1}{pqr,h_{i\ell j}} \sum_{m=0}^{p_i} \sum_{n=0}^{q_\ell} \sum_{k=0}^{r_j}$
$a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$
$\frac{1}{pqr,h_{i\ell j}} \sum_{m=0}^{p_{i-1}} \sum_{n=0}^{q_{\ell-1}} \sum_{k=0}^{r_{i-1}}$
$a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$
$\frac{1}{pqr,h_{i\ell j}} \sum_{m=m_i+0}^{p_i} \sum_{n=0}^{q_{\ell-1}} \sum_{k=0}^{r_{j-1}}$
$a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} -$
$\frac{1}{pqr,h_{i\ell j}} \sum_{k=k_j+0}^{r_j} \sum_{n=n_{\ell-0}+0}^{q_\ell} \sum_{m=1}^{p_{k-1}}$
$a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$
$= \frac{m_i n_\ell k_j}{pqr,h_{i\ell j}} \left(\frac{1}{m_i n_\ell k_j} \sum_{m=0}^{p_i} \sum_{n=0}^{q_\ell} \sum_{k=0}^{r_j} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$
$\frac{m_{k-1} n_{\ell-1} k_{j-1}}{pqr,h_{i\ell j}}$
$\left(\frac{1}{m_{i-1} n_{\ell-1} kj-1} \sum_{m=0}^{p_{i-0}} \sum_{n=0}^{q_{\ell-0}} \sum_{k=0}^{r_{j-0}} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$
$- \quad \frac{k_{j-1}}{pqr,h_{i\ell j}} \left(\frac{1}{k_{j-1}} \sum_{m=m_{i-0}+0}^{p_i} \sum_{n=0}^{q_{\ell-0}} \sum_{k=0}^{r_j} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$
$- \quad \frac{n_{\ell-1}}{pqr,h_{i\ell j}} \left(\frac{1}{n_{\ell-1}} \sum_{m=m_{k-0}+0}^{p_k} \sum_{n=0}^{q_{\ell-0}} \sum_{k=0}^{r_j} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$
$\frac{m_{k-1}}{pqr,h_{i\ell j}}$
$\left(\frac{1}{m_{k-1}} \sum_{k=0}^{r_j} \sum_{n=n_{\ell-0}+0}^{q_\ell} \sum_{m=0}^{p_{k-0}} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right).$

Since $x \in \sigma^{\chi^3_f}(P)$ the last three terms tend to zero uniformly in m, n, k in the sense, thus, for each r, s and u

$B_{i,\ell,j} = \frac{m_i n_\ell k_j}{pqr,h_{i\ell j}} \left(\frac{1}{m_i n_\ell k_j} \sum_{m=0}^{p_i} \sum_{n=0}^{q_\ell} \sum_{k=0}^{r_j} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) -$
$\frac{m_{i-1} n_{\ell-1} k_{j-1}}{pqr,h_{i\ell j}}$
$\left(\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=0}^{p_{i-0}} \sum_{n=0}^{q_{\ell-0}} \sum_{k=0}^{r_{j-0}} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right) +$
$O(1).$

Since $h_{i\ell j} = m_i n_\ell k_j - m_{i-1} n_{\ell-1} k_{j-1}$ we are granted for each i, ℓ and j the following

$$\frac{m_i n_\ell kj}{pqr,h_{i\ell j}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{m_{i-0} n_{\ell-0} k_{j-0}}{pqr,h_{i\ell j}} \leq \frac{1}{\delta}.$$

The terms
$\left(\frac{1}{m_i n_\ell k_j} \sum_{m=0}^{p_i} \sum_{n=0}^{q_\ell} \sum_{k=0}^{r_j} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$ and
$\left(\frac{1}{m_{i-1} n_{\ell-1} k_{j-1}} \sum_{m=0}^{p_{i-0}} \sum_{n=0}^{q_{\ell-0}} \sum_{k=0}^{r_{j-0}} a^{pqr}_{mnk} f\left[\left((m+n+k)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$ are both gai sequences for all i, ℓ and j. Thus $B_{i\ell j}$ is a gai sequence for each i, ℓ and j. Hence $x \in \sigma^{\chi^3_f}\left(A\sigma_{pqr} C_{\theta_{i,\ell,j}}, P\right)$.

4.4. Theorem

Let $\theta_{i,\ell,j} = \{m, n, k\}$ be a triple lacunary sequence with $limsup_\eta q_\eta < \infty$ and $limsup_i \overline{q_i} < \infty$ then for any Orlicz function f, $\sigma^{\chi^3_f}\left(A\sigma_{pqr} C_{\theta_{i,\ell,j}}, P\right) \subset \sigma^{\chi^3_f(p)}$.

Proof: Since $limsup_i q_i < \infty$ and $limsup_i \overline{q_i} < \infty$ there exists $H > 0$ such that

$q_i < H$, $\overline{q_\ell} < H$ and $q_j < H$ for all i,ℓ and j. Let $x \in \sigma^{\chi_f^3}\left(A\sigma_{pqr}C_{\theta_{i,\ell,j}}, P\right)$. Also there exist $i_0 > 0, \ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$ $b \geq \ell_0$ and $c \geq j_0$ and i,ℓ and j.

$$A'_{abc} = \frac{1}{pqr,h_{abc}} \sum_{m=0,m\in I_{a,b,c}}^{p} \sum_{n=0,n\in I_{a,b,c}}^{q} \sum_{k=0,k\in I_{a,b,c}}^{r} a_{mnk}^{pqr} f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}\right]^{p_{mnk}} \to 0\, as\, m,n,k \to \infty.$$

Let $G' = max\left\{A'_{a,b,c} : 1 \leq a \leq i_0,\ \ 1 \leq b \leq \ell_0 \ \ and\ \ 1 \leq c \leq j_0\right\}$ and p,q and r be such that $m_{i-1} < p \leq m_i$, $n_{\ell-1} < q \leq n_\ell$ and $m_{j-1} < r \leq m_j$. Thus we obtain the following:

$$\frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=1}^{r} a_{mnk}^{pqr} f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$\leq \frac{1}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{m=0}^{p_i} \sum_{n=0}^{q_\ell} \sum_{k=0}^{r_j} a_{mnk}^{pqr} f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$\leq \frac{1}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$$
$$\left(\sum_{m=0,m\in I_{a,b,c}}^{p} \sum_{n=0,n\in I_{a,b,c}}^{q} \sum_{k=0,k\in I_{a,b,c}}^{r} a_{mnk}^{pqr} f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$
$$= \frac{1}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c}A'_{a,b,c} + \frac{1}{m_{k-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq j)} h_{a,b,c}A'_{a,b,c}$$
$$\leq \frac{G'}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} h_{a,b,c} + \frac{1}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq J)} h_{a,b,c}A'_{a,b,c}$$
$$\leq \frac{G' m_{i_0}n_{\ell_0}k_{j_0}i_0\ell_0 j_0}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} + \frac{1}{pqr,m_{i-1}n_{\ell-1}j_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq j)} h_{a,b,c}A'_{a,b,c}$$
$$\leq \frac{G' m_{i_0}n_{\ell_0}k_{j_0}i_0\ell_0 j_0}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} + \left(sup_{a\geq i_0 \bigcup b \geq \ell_0 \bigcup c \geq j_0} A'_{a,b,c}\right) \frac{1}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq j)} h_{a,b,c}$$
$$\leq \frac{G' m_{i_0}n_{\ell_0}k_{j_0}i_0\ell_0 j_0}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} + \frac{\epsilon}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq \ell)\bigcup(j_0<c\leq j)} h_{a,b,c}$$
$$\leq \frac{G' m_{i_0}n_{\ell_0}k_{j_0}i_0\ell_0 j_0}{pqr,m_{i-1}n_{\ell-1}k_{j-1}} + \epsilon H^3.$$

Since m_i, n_ℓ and k_j both approaches infinity as both p,q and r approaches infinity, it follows that

$$\frac{1}{pqr} \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{k=0}^{r} a_{mnk}^{pqr} f\left[\left((m+n+k)!\,|x_{m+i,n+\ell,k+j}|\right)^{1/m+n+k}\right]^{p_{mnk}} = 0,\ uniformly\, in\, i, \ell\, and\, j.$$

Hence $x \in \sigma^{\chi_f^3}(P)$.

4.5. Theorem

Let $\theta_{i,\ell,j}$ be a triple lacunary sequence then

(i) $(x_{mnk}) \xrightarrow{P} \sigma^{\chi^3} C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$

(ii) $\left(A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right)$ is a proper subset of $C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$

(iii) If $x \in \Lambda^3$ and $(x_{mnk}) \xrightarrow{P} \sigma^{\chi^3} C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ then $(x_{mnk}) \xrightarrow{P} \sigma^{\chi^3}\left(A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right)$

(iv) $\sigma^{\chi^3} C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right) \bigcap \Lambda^3 = \sigma^{\chi^3}\left[A\sigma_{pqr}C_{\theta_{i,,\ell,j}}\right] \bigcap \Lambda^3.$

Proof: **(i)** Since for all i,ℓ and j

$$\left|\left\{(m,n,k) \in I_{i,\ell,j} : \left((m+n+k)!\,|x_{m+i,n+\ell,k+j} - 0|\right)^{1/m+n+k}\right\} = 0\right|$$
$$\leq \sum_{m=0,m\in I_{i,\ell,j}}^{p} \sum_{n=0,n\in I_{i,\ell,j}}^{q} \sum_{k=0,k\in I_{i,\ell,j}\ and\ |x_{m+i,n+\ell,k+j}|=0}^{r}$$
$$\left((m+n+k)!\,|x_{m+i,n+\ell,k+j} - 0|\right)^{1/m+n+k}$$
$$\leq \sum_{m=0,m\in I_{i,\ell,j}}^{p} \sum_{n=0,n\in I_{i,\ell,j}}^{q} \sum_{k=0,k\in I_{i,\ell,j}}^{r} \left((m+n+k)!\,|x_{m+i,n+\ell,k+j} - 0|\right)^{1/m+n+k},$$ for

all i,ℓ and j
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}$
$((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}=0$
This implies that for all i,ℓ and j

$$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i,\ell,j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}=0\right\}\right|=0.$$

(ii) let $x=(x_{mnk})$ be defined as follows:

$$(x_{mnk})=\begin{bmatrix} 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ . \\ . \\ . \\ 1 & 2 & 3 & \ldots\frac{\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!} & 0 & \ldots\\ . \\ . \\ . \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots\\ . \\ . \\ . \end{bmatrix};$$

Here x is an trible sequence and for all i,ℓ and j
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{k,\ell,j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}=0\right\}\right|=$

$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i,\ell,j}}\left(\frac{(m+n+k)!\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k}=0.$

Therefore $(x_{mnk})\xrightarrow{P}\sigma^{\chi^3}C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$. Also
$P-lim_{i,\ell,j}\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}((m+n+k)!\,|x_{m+i,n+\ell,k+j}|)^{1/m+n+k}=$

$P-\frac{1}{2}\left(lim_{i,\ell,j}\frac{1}{pqr,h_{i,\ell,j}}\left(\frac{(m+n+k)!\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{pqr,h_{i,\ell,j}}\right]^{m+n+k}}{(m+n+k)!}\right)^{1/m+n+k}+1\right)=$
$\frac{1}{2}$.

Therefore $(x_{mnk})\overset{P}{\nrightarrow}\sigma^{\chi^3}\left(A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right)$.

(iii) If $x\in\Lambda^3$ and $(x_{mnk})\xrightarrow{P}\sigma^{\chi^3}C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ then $(x_{mnk})\xrightarrow{P}\sigma^{\chi^3}\left(A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right)$.

Suppose $x\in\Lambda^3$ then for all i,ℓ and j, $((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}\le M$ for all m,n,k. Also for given $\epsilon>0$ and i,ℓ and j large for all i,ℓ and j we obtain the following:
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}=$

$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}\,and\,|x_{m+i,n+\ell,k+j}|\ge 0}^{r}$
$((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}+$
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k\in I_{i,\ell,j}\,and\,|x_{m+i,n+\ell,k+j}|\le 0}^{r}$

$((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}$
$\leq \frac{M}{pqr,h_{i\ell j}}\left|\left\{(m,n,k)\in I_{i,\ell,j}:((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}\right\}=0\right|+\epsilon.$
Therefore $x\in\Lambda^3$ and $(x_{mnk})\xrightarrow{P}\sigma^{\chi^3}C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$ then $(x_{mnk})\xrightarrow{P}\sigma^{\chi^3}\left(A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right)$.
(iv)$\sigma^{\chi^3}C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)\bigcap\Lambda^3=\sigma^{\chi^3}\left[A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right]\bigcap\Lambda^3$. follows from (i),(ii) and (iii).

4.6. Theorem

If f be any Orlicz function then $\sigma^{\chi^3_f}\left[A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right]\notin\sigma^{\chi^3}C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$

Proof: Let $x\in\sigma^{\chi^3_f}\left[A\sigma_{pqr}C_{\theta_{i,\ell,j}}\right]$, for all i,ℓ and j.
Therefore we have
$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}}^{r}a_{mnk}^{pqr}f\left[((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}\right]\geq$

$\frac{1}{pqr,h_{i\ell j}}\sum_{m=0,m\in I_{i,\ell,j}}^{p}\sum_{n=0,n\in I_{i,\ell,j}}^{q}\sum_{k=0,k\in I_{i,\ell,j}\ and\ |x_{m+r,n+s,k+u}|=0}^{r}$
$a_{mnk}^{pqr}f\left[((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}\right]>$
$\frac{1}{pqr,h_{i\ell j}}a_{mnk}^{pqr}f(0)\left|\left\{(m,n,k)\in I_{i,\ell,j}:((m+n+k)!\,|x_{m+i,n+\ell,k+j}-0|)^{1/m+n+k}\right\}=0\right|.$
Hence $x\notin\sigma^{\chi^3}C_{111}\left(\widehat{S_{\theta_{i,\ell,j}}}\right)$.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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