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A new extension of some well known fixed point theorems in partially ordered metric spaces

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Abstract In this paper we obtain fixed point results of weak Kannan and chatterjea type contractions using several control functions of one and two variables. The results are obtained in partially ordered metric spaces. There are two main theorems which have a number of corollaries. Illustrative examples are cited to show that the theorems actually contain their corollaries. The results are obtained without any assumption of continuity on the function. Some existing results in the literature are extended by our results. The methodology is a blending of order theoretic and analytic approaches.

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1. Introduction and Mathematical Preliminaries

Fixed point theory is one of the most important topics in the development of nonlinear analysis. Also, fixed point theory has been used effectively in many other branch of science, such as chemistry, biology, economics, computer science, engineering etc. It is a long time many mathematicians have studied on fixed point theory developing very useful results in this area. Now, we briefly recall some of those results.

Definition 1.1 (contraction)**.** A mapping $T : X \to X$, where $(X,\ d)$ is a metric space, is said to be a contraction if there exists $k \in [0,\ 1)$ such that

$$d(Tx,\ Ty) \le k\ d(x,\ y), \quad \text{for all}\ \ x,\ y \in X. \tag{1.1}$$

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Theorem 1.2 (Banach contraction mapping principle [3]). *Let $(X,\ d)$ be a complete metric space and $T : X \longrightarrow X$. If the mapping T satisfies (1.1), then T has a unique fixed point in X.*

Inequality (1.1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

Kannan [16, 17] established the following result in which the above question has been answered in the affirmative.

Theorem 1.3 (Kannan [16, 17]). *Let $(X,\ d)$ be a complete metric space and $T : X \to X$. If the mapping T satisfies the inequality*

$$d(Tx,\ Ty) \leq k \Big[d(x,\ Tx) + d(y,\ Ty)\Big], \ \text{where} \ \ k \in \Big[0,\ \frac{1}{2}\Big) \ \text{and} \ x,\ y \in X, \quad (1.2)$$

then T has a unique fixed point in X.

The mappings satisfying (1.2) are called Kannan type mappings.

A similar contractive condition has been introduced by Chatterjea [5] as following:

Theorem 1.4 (Chatterjea [5]). *Let $(X,\ d)$ be a complete metric space and $T : X \to X$. If the mapping T satisfies the inequality*

$$d(Tx,\ Ty) \leq k \Big[d(x,\ Ty) + d(y,\ Tx)\Big], \ \text{where} \ \ k \in \Big[0,\ \frac{1}{2}\Big) \ \text{and} \ x,\ y \in X, \quad (1.3)$$

then T has a unique fixed point in X.

The mappings satisfying (1.3) are called Chatterjea type mapping or C-contraction.

Weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber et al. in Hilbert spaces [1] and subsequently extended to metric spaces by Rhoades [28]. We state the result of Rhoades [28] in the following.

Definition 1.5 (Weakly contractive mapping [28]). A mapping $T : X \longrightarrow X$, where $(X,\ d)$ is a metric space, is said to be weakly contractive if for $x,\ y \in X$,

$$d(Tx,\ Ty) \leq d(x,\ y) - \varphi\Big(d(x,\ y)\Big), \quad (1.4)$$

where $\varphi : [0,\ \infty) \longrightarrow [0,\ \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

If one takes $\varphi(t) = (1 - k)\ t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

Theorem 1.6 (Rhoades [28]). *Let $(X,\ d)$ be a complete metric space and $T : X \to X$ a weakly contractive map. Then T has a unique fixed point in X.*

Choudhury [7] introduced a generalization of Chatterjea type contraction as follows:

Definition 1.7 (weakly C - contractive [7]). A mapping $T : X \longrightarrow X$, where $(X,\ d)$ is a metric space, is said to be weakly C - contractive (or weak Chatterjea type contraction) if

$$d(Tx,\ Ty) \leq \frac{1}{2} \Big[d(x,\ Ty) + d(y,\ Tx)\Big] - \varphi\Big(d(x,\ Ty),\ d(y,\ Tx)\Big), \ \text{for} \ x,\ y \in X, \quad (1.5)$$

where $\varphi : [0,\ \infty) \times [0,\ \infty) \longrightarrow [0,\ \infty)$ is a continuous function such that $\varphi(x,\ y) = 0$ if and only if $x = y = 0$.

Choudhury [7] showed that a mapping satisfying the inequality (1.5) have unique fixed point in complete metric space and proved that there is no requirement of continuity of the C - contraction. Also, (1.5) reduces to (1.3) if one takes $\varphi(x,\ y) = (\frac{1}{2} - k)\ (x + y)$ where $k \in [\frac{1}{2},\ 1)$.

Dutta and Choudhury [13] introduced a new generalization of the Banach contraction mapping principle. This new generalization is more general than the concept of Rhoades in [28]. The result of Dutta and Choudhury is following:

Theorem 1.8 (Dutta and Choudhury [13]). *Let $(X,\ d)$ be a complete metric space and $T : X \to X$ be a self-mapping satisfying the inequality*

$$\psi\Big(d(Tx,\ Ty)\Big) \leq \psi\Big(d(x,\ y)\Big) - \varphi\Big(d(x,\ y)\Big),\ \text{for } x,\ y \in X, \tag{1.6}$$

where, $\psi,\ \varphi : [0,\ \infty) \longrightarrow [0,\ \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Weak contractive and weak contractive type conditions have been used by many researchers to establish fixed point results in metric spaces [2, 4, 6, 8, 9, 11, 12, 20, 21, 24–26].

In recent years fixed point theory has experienced a rapid development in partially ordered metric spaces. An early result in this direction is due to Turinici [29] in which fixed point problems were studied in partially ordered uniform spaces. Later, this branch of fixed point theory has developed through a number of works some of which are in [10, 14, 15, 18, 19, 22, 23, 27]. Particularly, Harjani et. al have established a generalized weak contraction principle in partially ordered metric spaces [14].

Definition 1.9. Let $(X,\ \preceq)$ be a partially ordered set and $T : X \longrightarrow X$. The mapping T is said to be nondecreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $Tx_1 \preceq Tx_2$ and nonincreasing if for all $x_1,\ x_2 \in X$, $x_1 \preceq x_2$ implies $Tx_1 \succeq Tx_2$.

In our results in the following sections we will use the following class of functions.
We denote by Ψ the set of all functions $\psi\colon [0,\ \infty) \longrightarrow [0,\ \infty)$ satisfying
(i_ψ) ψ is continuous and monotone non-decreasing,
(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$;
by Φ we denote the set of all functions $\alpha\colon [0,\ \infty) \longrightarrow [0,\ \infty)$ such that
(i_α) α is bounded on any bounded interval in $[0,\ \infty)$,
(ii_α) α is continuous at 0 and $\alpha(0) = 0$;
and by Θ we denote the set of all functions $\beta\colon [0,\ \infty) \times [0,\ \infty) \longrightarrow [0,\ \infty)$ such that
(i_β) β is bounded on any bounded subset of $[0,\ \infty) \times [0,\ \infty)$,
(ii_β) β is continuous at $(0,\ 0)$ and $\beta(0,\ 0) = 0$.

2. Main Results

Lemma 2.1. *Let $(X,\ d)$ be a metric space. Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_{n+1},\ x_n) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a Cauchy sequence in $(X,\ d)$, then there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four *sequences tend to* ϵ *when* $k \longrightarrow \infty$ *:*

$$\Big\{d(x_{m(k)}, x_{n(k)})\Big\},\ \Big\{d(x_{m(k)}, x_{n(k)+1})\Big\},\ \Big\{d(x_{n(k)}, x_{m(k)+1})\Big\},\ \Big\{d(x_{m(k)+1}, x_{n(k)+1})\Big\}. \tag{2.2}$$

Proof. Suppose that $\{x_n\}$ is a sequence in $(X,\ d)$ satisfying (2.1) which is not Cauchy. Then there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(x_{m(k)},\ x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer, we get

$$n(k) > m(k) > k,\ d(x_{m(k)},\ x_{n(k)}) \geq \epsilon \ \text{ and } \ d(x_{m(k)},\ x_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{n(k)}) < \epsilon + d(x_{n(k)-1},\ x_{n(k)}).$$

Letting $k \longrightarrow \infty$ in the above inequality and using (2.1), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon. \tag{2.3}$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)+1}) \leq d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Letting $k \longrightarrow \infty$ in the above inequalities and using (2.1) and (2.3), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon. \tag{2.4}$$

That the remaining two sequences in (2.2) tend to ϵ can be proved in a similar way. ∎

Theorem 2.2. *Let* $(X,\ \preceq)$ *be a partially ordered set and suppose that there is a metric* d *on* X *such that* $(X,\ d)$ *is a complete metric space. Let* $T : X \longrightarrow X$ *be a nondecreasing mapping with respect to* $\preceq$*. Suppose that there exist* $\psi \in \Psi$, $\varphi \in \Phi$ *and* $\theta \in \Theta$ *such that*

$$\psi(x) \leq \varphi(y) \implies x \leq y, \text{ for } x,\ y \in [0,\ \infty), \tag{2.5}$$

for any sequence $\{x_n\}$ *in* $[0,\ \infty)$ *with* $x_n \longrightarrow t > 0$,

$$\psi(t) - \overline{\lim}\ \varphi\Big(\frac{1}{2}\ (x_n + x_{n+1})\Big) + \underline{\lim}\ \theta(x_n,\ x_{n+1}) > 0, \tag{2.6}$$

and for all $x,\ y \in X$ *with* $x \preceq y$,

$$\psi\Big(d(Tx,\ Ty)\Big) \leq \varphi\Big(\frac{1}{2}\ \Big[d(x,\ Tx) + d(y,\ Ty)\Big]\Big) - \theta\Big(d(x,\ Tx),\ d(y,\ Ty)\Big). \tag{2.7}$$

Also suppose that
(a) T *is continuous or*
(b) X *has the following properties:*

(i) *if a nondecreasing sequence* $\{x_n\} \longrightarrow x$, *then* $x_n \preceq x$, *for all* $n \geq 0$;

(ii) *if a nonincreasing sequence* $\{y_n\} \longrightarrow y$, *then* $y \preceq y_n$, *for all* $n \geq 0$.

If there exists $x_0 \in X$ *such that* $x_0 \preceq Tx_0$, *then* T *has a fixed point in* X.

Proof. Starting with x_0, we construct the sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n, \text{ for all } n \geq 0. \tag{2.8}$$

Since T is non-decreasing and $x_0 \preceq Tx_0$, we have

$$x_0 \preceq Tx_0 = x_1 \preceq Tx_1 = x_2 \preceq ... \preceq Tx_{n-1} = x_n \preceq Tx_n = x_{n+1} \preceq .... \tag{2.9}$$

Let $R_n = d(x_n,\ x_{n+1})$, for all $n \geq 0$.
Since $x_n \preceq x_{n+1}$, from (2.7) and (2.8), we have

$$\begin{aligned}\psi(R_{n+1}) &= \psi\Big(d(x_{n+1},\ x_{n+2})\Big) = \psi\Big(d(Tx_n,\ Tx_{n+1})\Big)\\ &\leq \varphi\Big(\frac{1}{2}\ \Big(d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})\Big)\Big) - \theta\Big(d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\Big)\\ &\leq \varphi\Big(\frac{1}{2}\ (R_n + R_{n+1})\Big) - \theta(R_n,\ R_{n+1}),\end{aligned} \tag{2.10}$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi\Big(\frac{1}{2}\ (R_n + R_{n+1})\Big)$, which by (2.5), implies that $R_{n+1} \leq \frac{1}{2}\ (R_n+R_{n+1})$, that is, $R_{n+1} \leq R_n$. Thus, $\{R_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists an $r \geq 0$ such that

$$R_n = d(x_n,\ x_{n+1}) \longrightarrow r \ \text{ as } \ n \longrightarrow \infty. \tag{2.11}$$

Taking limit supremum in both sides of (2.10), using (2.11), the continuity of ψ and the properties of φ and θ, we obtain

$$\psi(r) \leq \overline{\lim}\ \varphi\Big(\frac{1}{2}\ (R_n + R_{n+1})\Big) + \overline{\lim}\ \Big(-\ \theta(R_n,\ R_{n+1})\Big).$$

Since $\overline{\lim}\ \Big(-\ \theta(R_n,\ R_{n+1})\Big) = -\ \underline{\lim}\ \theta(R_n,\ R_{n+1})$, it follows that

$$\psi(r) \leq \overline{\lim}\ \varphi\Big(\frac{1}{2}\ (R_n + R_{n+1})\Big) - \underline{\lim}\ \theta(R_n,\ R_{n+1}),$$

that is,

$$\psi(r) - \overline{\lim}\ \varphi\Big(\frac{1}{2}\ (R_n + R_{n+1})\Big) + \underline{\lim}\ \theta(R_n,\ R_{n+1}) \leq 0,$$

which, by (2.6) and (2.11), is a contradiction unless $r = 0$. Hence, we have

$$R_n = d(x_n,\ x_{n+1}) \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty. \tag{2.12}$$

Next we show that $\{x_n\}$ is a Cauchy sequence.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by the Lemma 2.1, there exists an $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for all positive integers k, $n(k) > m(k) > k$ and

$$\lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1}) = \epsilon. \tag{2.13}$$

As $n(k) > m(k)$, $x_{m(k)} \preceq x_{n(k)}$. Applying (2.7) and using (2.8), we have

$$\begin{aligned}\psi\Big(d(x_{m(k)+1}, x_{n(k)+1})\Big) &= \psi\Big(d(Tx_{m(k)}, Tx_{n(k)})\Big)\\ &\leq \varphi\Big(\frac{1}{2}\ \Big(d(x_{m(k)},\ x_{m(k)+1}) + d(x_{n(k)},\ x_{n(k)+1})\Big)\Big)\end{aligned}$$

$$-\theta\Big(d(x_{m(k)},\ x_{m(k)+1}),\ d(x_{n(k)},\ x_{n(k)+1})\Big),$$

which, in view of the fact that $\theta \geq 0$, yields

$$\psi\Big(d(x_{m(k)+1}, x_{n(k)+1})\Big) \leq \varphi\Big(\frac{1}{2}\ \Big(d(x_{m(k)},\ x_{m(k)+1}) + d(x_{n(k)},\ x_{n(k)+1})\Big)\Big),$$

which, by (2.5), implies that

$$d(x_{m(k)+1},\ x_{n(k)+1}) \leq \frac{1}{2}\ \Big(d(x_{m(k)},\ x_{m(k)+1}) + d(x_{n(k)},\ x_{n(k)+1})\Big).$$

Taking limit $k \longrightarrow \infty$ in the above inequality using (2.12) and (2.13), we obtain $\epsilon \leq 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $z \in X$ such that

$$x_n \longrightarrow z \quad \text{as} \quad n \longrightarrow \infty. \tag{2.14}$$

Let the condition (a) holds.
The continuity of T implies that $Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$, that is, z is a fixed point of T.

Next we suppose that the condition (b) holds.
By (2.9) and (2.14), we have $x_n \preceq z$ for all $n \geq 0$. Then applying (2.7) and using (2.8), we have

$$\begin{aligned}\psi\Big(d(x_{n+1},\ Tz)\Big) &= \psi\Big(d(Tx_n,\ Tz)\Big)\\ &\leq \varphi\Big(\frac{1}{2}\ \Big(d(x_n,\ x_{n+1}) + d(z,\ Tz)\Big)\Big) - \theta\Big(d(x_n,\ x_{n+1}),\ d(z,\ Tz)\Big),\end{aligned}$$

which, in view of the fact that $\theta \geq 0$, yields

$$\psi\Big(d(x_{n+1},\ Tz)\Big) \leq \varphi\Big(\frac{1}{2}\ \Big(d(x_n,\ x_{n+1}) + d(z,\ Tz)\Big)\Big),$$

which, by (2.5), implies that

$$d(x_{n+1},\ Tz) \leq \frac{1}{2}\ \Big(d(x_n,\ x_{n+1}) + d(z,\ Tz)\Big).$$

Taking limit $n \longrightarrow \infty$ in the above inequality and using (2.14), we have

$$d(z,\ Tz) \leq \frac{1}{2}\ d(z,\ Tz),$$

which implies that $d(Tz,\ z) = 0$, that is, $z = Tz$, that is, z is a fixed point of T. ∎

Considering ψ to be the identity mapping and $\theta(x,\ y) = 0$ for all $(x,\ y) \in [0,\ \infty) \times [0,\ \infty)$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\varphi \in \Phi$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\overline{\lim}\ \varphi\Big(\frac{1}{2}\ (x_n + x_{n+1})\Big) < t$ and for all $x,\ y \in X$ with $x \preceq y$,*

$$d(Tx,\ Ty) \leq \varphi\Big(\frac{1}{2}\ \Big[d(x,\ Tx) + d(y,\ Ty)\Big]\Big). \tag{2.15}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering φ to be identical with the function ψ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \theta(x_n, x_{n+1}) > 0$ and for all $x, y \in X$ with $x \preceq y$,*

$$\psi\Big(d(Tx, Ty)\Big) \leq \psi\Big(\frac{1}{2}\ \Big[d(x, Tx)+d(y, Ty)\Big]\Big) - \theta\Big(d(x, Tx), d(y, Ty)\Big). \tag{2.16}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

If ψ and φ are the identity mappings in Theorem 2.2, we have the following corollary.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\theta \in \Theta$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \theta(x_n, x_{n+1}) > 0$ and for all $x, y \in X$ with $x \preceq y$,*

$$d(Tx, Ty) \leq \frac{1}{2}\ \Big(d(x, Tx) + d(y, Ty)\Big) - \theta\Big(d(x, Tx), d(y, Ty)\Big). \tag{2.17}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering ψ and φ to be the identity mappings and $\theta(x, y) = (\frac{1}{2} - k)\ (x+y)$, where $0 \leq k < \frac{1}{2}$ in Theorem 2.2, we have the following corollary.

Corollary 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Assume that there exists $k \in [0, \frac{1}{2})$ such that for all $x, y \in X$ with $x \preceq y$,*

$$d(Tx, Ty) \leq k\ \Big[d(x, Tx) + d(y, Ty)\Big]. \tag{2.18}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering $\theta \in \Theta$ to be defined as $\theta(x, y) = \phi\Big(\frac{x+y}{2}\Big)$, where $\phi \in \Phi$ in Theorem 2.2, we have the following corollary.

Corollary 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\varphi, \phi \in \Phi$ such that (2.5) is satisfied, for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \longrightarrow t > 0$,*

$$\psi(t) - \overline{\lim}\ \varphi(x_n) + \underline{\lim}\ \phi(x_n) > 0, \tag{2.19}$$

and for all $x, y \in X$ with $x \preceq y$,

$$\psi\Big(d(Tx, Ty)\Big) \leq \varphi\Big(\frac{1}{2}\ [d(x, Tx) + d(y, Ty)]\Big) - \phi\Big(\frac{1}{2}\ [d(x, Tx) + d(y, Ty)]\Big). \tag{2.20}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Example 2.8. Let $X = \{0,\ 1,\ 2,\ 3,\ ...\}$. We define a partial ordering '$\preceq$' in X as $x \preceq y$ if and only if $x \geq y$. We define the metric d on X as

$$d(x,\ y) = \begin{cases} x + y, \text{ if } x \neq y, \\ 0, \text{ if } \ x = y. \end{cases}$$

Let $T : X \to X$ be defined as follows

$$Tx = \begin{cases} x - 1, & \text{if } x \geq 1, \\ 0, & \text{if } \ x = 0. \end{cases}$$

Let $\psi,\ \varphi : [0, \infty) \to [0, \infty)$ be defined as follows:

$$\psi(t) = t^2, \quad \varphi(t) = \begin{cases} t^2, & \text{for } t \geq \frac{1}{2}, \\ \frac{t^2}{2}, & \text{for } 0 \leq t < \frac{1}{2} \end{cases}$$

Let $\theta : [0,\ \infty)^2 \longrightarrow [0,\ \infty)$ be defined as follows:
for $(s,\ t) \in [0,\ \infty)^2$ with $u = \max\{s,\ t\}$,

$$\theta(s,\ t) = \begin{cases} \frac{1}{4}, & \text{if } \ u \geq \frac{1}{2}, \\ \frac{u^2}{4}, & \text{if } \ u < \frac{1}{2}. \end{cases}$$

For $x_0 = 10$, $Tx_0 = 9$. Then we have $x_0 \preceq Tx_0$.
Now, we will verify that (2.7) is satisfied for all $x,\ y \in X$ with $x \preceq y$. Without loss of generality, we assume that $x \geq y$ and discuss the following cases. Then the following cases are possible.

Case I $x \neq y$ and $y \neq 0$, then

$$\psi\Big(d(Tx,\ Ty)\Big) = \psi\Big(d(x-1,\ y-1)\Big) = \psi(x+y-2) = (x+y-2)^2,$$

$$\varphi\Big(\frac{d(x,\ Tx) + d(y,\ Ty)}{2}\Big) = \varphi\Big(\frac{2x-1+2y-1}{2}\Big) = (x+y-1)^2$$

and

$$\theta\Big(d(x,\ Tx),\ d(y,\ Ty)\Big) = \theta(2x-1,\ 2y-1) = \frac{1}{4}.$$

Case II $x \neq y$ and $y = 0$, then

$$\psi\Big(d(Tx,\ Ty)\Big) = \psi\Big(d(x-1,\ 0)\Big) = \psi(x-1) = (x-1)^2,$$

$$\varphi\Big(\frac{d(x,\ Tx) + d(y,\ Ty)}{2}\Big) = \varphi\Big(\frac{2x-1+0}{2}\Big) = \Big(x - \frac{1}{2}\Big)^2$$

and

$$\theta\Big(d(x,\ Tx),\ d(y,\ Ty)\Big) = \theta(2x-1,\ 0) = \frac{1}{4}.$$

Cases III $x = y$ and $y \neq 0$, then

$$\psi\Big(d(Tx,\ Ty)\Big) = 0,$$

$$\varphi\Big(\frac{d(x,\ Tx) + d(y,\ Ty)}{2}\Big) = \varphi\Big(d(x,\ Tx)\Big) = \varphi(2x-1) = (2x-1)^2$$

and

$$\theta\Big(d(x,\ Tx),\ d(y,\ Ty)\Big) = \theta\Big(d(x,\ Tx),\ d(x,\ Tx)\Big) = \theta(2x-1,\ 2x-1) = \frac{1}{4}.$$

Cases IV $x = y = 0$, then

$$\psi\Big(d(Tx,\ Ty)\Big) = 0,\ \ \varphi\Big(\frac{d(x,\ Tx) + d(y,\ Ty)}{2}\Big) = 0 \ \text{ and } \ \theta\Big(d(x,\ Tx),\ d(y,\ Ty)\Big) = 0.$$

In all the cases the inequality (2.7) is satisfied for all $x,\ y \in X$ with $x \preceq y$. Hence the required conditions of Theorem 2.2 are satisfied and it is seen that 0 is a fixed point of T.

Remark 2.9. In the above example, ψ is not the identity mapping and also φ is not identical with the function ψ. Therefore, Corollaries 2.3, 2.4, 2.5 and 2.6 are not applicable to this example and hence Theorem 2.2 properly contains its Corollaries 2.3, 2.4, 2.5 and 2.6.

Theorem 2.10. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\varphi,\ \phi \in \Phi$ such that (2.5) and (2.19) are satisfied and for all $x,\ y \in X$ with $x \preceq y$,*

$$\psi\Big(d(Tx,\ Ty)\Big) \le \varphi\Big(\frac{1}{2}\ \Big[d(x,\ Ty) + d(y,\ Tx)\Big]\Big) - \phi\Big(\frac{1}{2}\ \Big[d(x,\ Ty) + d(y,\ Tx)\Big]\Big). \tag{2.21}$$

Also suppose that either condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.2. Arguing similarly as in the proof of Theorem 2.2, we prove that the sequence $\{x_n\}$ satisfies (2.9).

Let $R_n = d(x_n,\ x_{n+1})$, for all $n \ge 0$.
Since $x_n \preceq x_{n+1}$, applying (2.21) and using (2.8), we have

$$\begin{aligned}
\psi(R_{n+1}) &= \psi(d(x_{n+1},\ x_{n+2})) = \psi(d(Tx_n,\ Tx_{n+1}))\\
&\le \varphi(\frac{1}{2}\ (d(x_n,\ x_{n+2}) + d(x_{n+1},\ x_{n+1}))) - \phi(\frac{1}{2}\ (d(x_n,\ x_{n+2}) + d(x_{n+1},\ x_{n+1})))\\
&\le \varphi(\frac{1}{2}\ d(x_n,\ x_{n+2})) - \phi(\frac{1}{2}\ d(x_n,\ x_{n+2})),
\end{aligned} \tag{2.22}$$

which, in view of the fact that $\phi \ge 0$, yields

$$\psi(R_{n+1}) \le \varphi\Big(\frac{1}{2}\ d(x_n,\ x_{n+2})\Big),$$

which, by (2.5), implies that

$$R_{n+1} \le \frac{1}{2}\ d(x_n,\ x_{n+2}) \le \frac{1}{2}\ \Big(d(x_n,\ x_{n+1}) + d(x_{n+1},\ x_{n+2})\Big) \le \frac{1}{2}\ (R_n + R_{n+1}) \tag{2.23}$$

that is, $R_{n+1} \le R_n$. Thus the sequence $\{R_n\}$ is monotone decreasing and satisfies (2.11), that is,

$$R_n = d(x_n,\ x_{n+1}) \longrightarrow r \ \text{ as } \ n \longrightarrow \infty.$$

Taking limit $n \longrightarrow \infty$ in (2.23) and using (2.11), we have

$$\frac{1}{2}\ d(x_n,\ x_{n+2}) = Q_n\ (\text{say}) \longrightarrow r \ \text{ as } \ n \longrightarrow \infty. \tag{2.24}$$

Taking limit supremum in both sides of (2.22), using (2.11), the continuity of ψ and the property of φ and ϕ, we obtain

$$\begin{aligned}\psi(r) &\le \overline{\lim}\ \varphi\Big(\frac{1}{2}\ d(x_n,\ x_{n+2})\Big) + \overline{\lim}\ (-\ \phi\Big(\frac{1}{2}\ d(x_n,\ x_{n+2})\Big)\Big) \\ &\le \overline{\lim}\ \varphi(Q_n) + \overline{\lim}\ \Big(-\ \phi(Q_n)\Big).\end{aligned}$$

Since $\overline{\lim}\ \Big(-\ \phi(Q_n)\Big) = -\ \underline{\lim}\ \phi(Q_n)$, it follows that

$$\psi(r) \le \overline{\lim}\ \varphi(Q_n) - \underline{\lim}\ \phi(Q_n),$$

that is,

$$\psi(r) - \overline{\lim}\ \varphi(Q_n) + \underline{\lim}\ \phi(Q_n) \le 0,$$

which, by (2.19) and (2.24), is a contradiction unless $r = 0$. Hence, (2.12) is satisfied, that is,

$$R_n = d(x_n,\ x_{n+1}) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Next we show that $\{x_n\}$ is a Cauchy sequence.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by the Lemma 2.1, there exists an $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for all positive integers k, $n(k) > m(k) > k$,

$$\lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)+1}) = \epsilon, \tag{2.25}$$

and (2.4) and (2.13) hold which are respectively,

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon,$$

and

$$\lim_{k\to\infty} d(x_{m(k)+1},\ x_{n(k)+1}) = \epsilon.$$

Let $S_k = \dfrac{1}{2}\ \Big(d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)},\ x_{m(k)+1})\Big)$. By (2.4) and (2.25), we have

$$\lim_{k\to\infty} S_k = \epsilon. \tag{2.26}$$

As $n(k) > m(k)$, $x_{m(k)} \preceq x_{n(k)}$. Applying (2.21) and using (2.8), we have

$$\begin{aligned}\psi\Big(d(x_{m(k)+1}, x_{n(k)+1})\Big) &= \psi\Big(d(Tx_{m(k)}, Tx_{n(k)})\Big) \\ &\le \varphi\Big(\frac{1}{2}\ \Big(d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)},\ x_{m(k)+1})\Big)\Big) \\ &\qquad - \phi\Big(\frac{1}{2}\ \Big(d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)},\ x_{m(k)+1})\Big)\Big) \\ &\le \varphi(S_k) - \phi(S_k).\end{aligned}$$

Taking limit supremum in the above inequality, using (2.13), the continuity of ψ and the property of φ and ϕ, we obtain

$$\psi(\epsilon) \le \overline{\lim}\ \varphi(S_k) + \overline{\lim}\ \Big(-\ \phi(S_k)\Big) \le \overline{\lim}\ \varphi(S_k) + \overline{\lim}\ \Big(-\ \phi(S_k)\Big).$$

Since $\overline{\lim}\ \Big(-\ \phi(S_k)\Big) = -\ \underline{\lim}\ \phi(S_k)$, it follows that

$$\psi(\epsilon) \le \overline{\lim}\ \varphi(S_k) - \underline{\lim}\ \phi(S_k),$$

that is,

$$\psi(\epsilon) - \overline{\lim}\ \varphi(S_k) + \underline{\lim}\ \phi(S_k) \le 0,$$

which, by (2.19) and (2.26), is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence and satisfies (2.14), that is,

$$x_n \longrightarrow z \quad \text{as} \quad n \longrightarrow \infty.$$

If the condition (a) holds, then arguing similarly as in the proof of the theorem 2.2, we prove that z is a fixed point of T.

Next we suppose that the condition (b) holds.
By (2.9) and (2.14), we have $x_n \preceq z$ for all $n \geq 0$. Then applying (2.21) and using (2.8), we have

$$\begin{aligned}\psi\Big(d(x_{n+1},\ Tz)\Big) &= \psi\Big(d(Tx_n,\ Tz)\Big)\\ &\leq \varphi\Big(\frac{1}{2}\ \Big(d(x_n,\ Tz)+d(z,\ x_{n+1})\Big)\Big)\\ &\qquad - \phi\Big(\frac{1}{2}\ \Big(d(x_n,\ Tz)+d(z,\ x_{n+1})\Big)\Big),\end{aligned}$$

which, in view of the fact that $\phi \geq 0$, yields

$$\psi\Big(d(x_{n+1},\ Tz)\Big) \leq \varphi\Big(\frac{1}{2}\ \Big(d(x_n,\ Tz)+d(z,\ x_{n+1})\Big)\Big),$$

which, by (2.5), implies that

$$d(x_{n+1},\ Tz) \leq \frac{1}{2}\ \Big(d(x_n,\ Tz)+d(z,\ x_{n+1})\Big).$$

Taking limit $n \longrightarrow \infty$ in the above inequality using (2.14), we have

$$d(z,\ Tz) \leq \frac{1}{2}\ d(z,\ Tz),$$

which implies that $d(Tz,\ z) = 0$, that is, $z = Tz$, that is, z is a fixed point of T. ∎

Considering ψ to be the identity mapping and $\phi(t) = 0$ for all $t \in [0,\ \infty)$ in Theorem 2.10, we have the following corollary.

Corollary 2.11. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\varphi \in \Phi$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\overline{\lim}\ \varphi(x_n) < t$ and for all $x,\ y \in X$ with $x \preceq y$,*

$$d(Tx,\ Ty) \leq \varphi\Big(\frac{1}{2}\ \Big[d(x,\ Ty)+d(y,\ Tx)\Big]\Big). \tag{2.27}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering φ to be identical with the function ψ in Theorem 2.10, we have the following corollary.

Corollary 2.12. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that for any sequence $\{x_n\}$ in $[0,\ \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \phi(x_n) > 0$ and for all $x,\ y \in X$ with $x \preceq y$,*

$$\psi\Big(d(Tx,\ Ty)\Big) \leq \psi\Big(\frac{1}{2}\ \Big[d(x,\ Ty)+d(y,\ Tx)\Big]\Big) - \phi\Big(\frac{1}{2}\ \Big[d(x,\ Ty)+d(y,\ Tx)\Big]\Big). \tag{2.28}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

If ψ and φ are the identity mappings in Theorem 2.10, we have the following corollary.

Corollary 2.13. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists $\phi \in \Phi$ such that for any sequence $\{x_n\}$ in $[0, \infty)$ with $x_n \longrightarrow t > 0$, $\underline{\lim}\ \phi(x_n) > 0$ and for all $x, y \in X$ with $x \preceq y$,*

$$d(Tx, Ty) \leq \frac{1}{2}\Big[d(x, Ty) + d(y, Tx)\Big] - \phi\Big(\frac{1}{2}\Big[d(x, Ty) + d(y, Tx)\Big]\Big). \tag{2.29}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Considering ψ and φ to be the identity mappings and $\phi(t) = (1-2k)\, t$, where $0 \leq k < \frac{1}{2}$ in Theorem 2.10, we have the following corollary.

Corollary 2.14. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping with respect to $\preceq$. Assume that there exists $k \in [0, \frac{1}{2})$ such that for all $x, y \in X$ with $x \preceq y$,*

$$d(Tx, Ty) \leq k\Big[d(x, Ty) + d(y, Tx)\Big]. \tag{2.30}$$

Also suppose that the condition (a) or (b) of Theorem 2.2 holds. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Example 2.15. Let $X = [0, 1]$ with the usual metric d. We define a partial ordering '$\preceq$' in X as $x \preceq y$ if and only if $x \geq y$.

Let $T : X \longrightarrow X$ be defined as follows:

$$Tx = \begin{cases} 0, & \text{if } \quad 0 \leq x \leq \dfrac{1}{2}, \\ \dfrac{1}{16}, & \text{if } \quad \dfrac{1}{2} < x \leq 1. \end{cases}$$

Let ψ, φ, ϕ: $[0, \infty) \longrightarrow [0, \infty)$ be given respectively by the formulas

$$\psi(t) = t^2, \quad \varphi(t) = \begin{cases} t^2, & \text{if } t \leq 1, \\ \dfrac{t^2}{2}, & \text{if } t > 1, \end{cases} \qquad \phi(t) = \begin{cases} \dfrac{t^2}{100}, & \text{if } t \leq 1, \\ t^2, & \text{if } t > 1. \end{cases}$$

It can be verified that (2.20) and (2.21) hold for all $x, y \in X$ with $x \preceq y$. Hence conditions of Corollary 2.7 and Theorem 2.10 are satisfied and it is seen that 0 is a fixed point of T.

Note 2.16. In the above example, the function T is not continuous.

Remark 2.17. In the above example, ψ is not the identity mapping and also φ is not identical with the function ψ. Therefore, Corollaries 2.11, 2.12, 2.13 and 2.14 are not applicable to this example and hence Theorem 2.10 properly contains its Corollaries 2.11, 2.12, 2.13 and 2.14.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Ya. I. Alber and S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, in: I. Gohberg, Yu. Lyubich (Eds.), New Results in Operator Theory, in: Advances and Appl. 98, Birkhuser, Basel, (1997) 7–22.

[2] G. V. R. Babu and P. D. Sailaja, Common Fixed Points of $(\psi,\ \varphi)$ Weak quasi contractions with property (E.A.), Int. J. Math. Sci Comput., 1 (2011) 29–37.

[3] S. Banach, Sur les oprations dans les ensembles abstraits et leurs applications aux quations intgrales, Fund Math., 3 (1922) 133–181.

[4] S. Chandok, Some common fixed point theorems for generalized f- weakly contractive mappings, J. Appl. Math. Informatics, 29 (2011) 257–265.

[5] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci., 25 (1972) 727–730.

[6] C. E. Chidume, H. Zegeye and S. J.Aneke, Approximation of fixed points of weakly contractive nonselfmaps in Banach spaces, J. Math. Anal. Appl., 270 (2002) 189–199.

[7] B. S. Choudhury, Unique fixed point theorem for weak C-contractive mappings, Kathmandu Univ. J. Sci. Eng. Tech., 5 (2009) 6–13.

[8] B. S. Choudhury, P. Konar, B. E. Rhoades and N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlinear Anal., 74(2011) 2116–2126.

[9] B. S. Choudhury and A. Kundu, (ψ,α,β)-weak contractions in partially ordered metric spaces, Appl. Math. Lett., 25 (2012) 6–10.

[10] Binayak S. Choudhury and N. Metiya, Fixed point theorems for almost contractions in partially ordered metric spaces, Ann. Univ. Ferrara, 58 (2012) 21–36.

[11] B. S. Choudhury, N. Metiya and M. Postolache, A generalizedweak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory Appl., 2013 (2013) 152.

[12] D. Dorić, Common fixed point for generalized (ψ,φ) - weak contractions, Appl. Math. Lett., 22 (2009) 1896–1900.

[13] P. N. Dutta and B. S. Choudhury, A generalisation of contraction principle in metric spaces, Fixed Point Theory Appl., 2008 (2008), Article ID 406368.

[14] J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal., 71 (2009) 3403–3410.

[15] J. Harjani and K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., 72 (2010) 1188–1197.

[16] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 60(1968) 71–76.

[17] R. Kannan, Some results of fixed points-II, Amer. Math. Monthly, 76 (1969) 405–408.

[18] E. Karapinar, Tripled fixed point theorems in partially ordered metric spaces, Stud. Univ. Babeş-Bolyai Math., 58(2013) 75–85.

[19] V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. TMA, 70 (2009) 4341–4349.

[20] S. Moradi and A. Farajzadeh, On the fixed point of (ψ,φ) - weak and generalized (ψ,φ) - weak contraction mappings, Appl. Math. Lett., 25 (2012) 1257–1262.

[21] H. K. Nashine and B. Samet, Fixed point results for mappings satisfying $(\psi,\ \phi)$-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal., 74 (2011) 2201–2209.

[22] H. K. Nashine, B. Samet and C. Vetro, Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces, Math. Comput. Modelling, 54

(12) (2011) 712–720.
[23] J. J. Nieto and R. López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22 (2005) 223–239.
[24] O. Popescu, Fixed points for (ψ, ϕ)- weak contractions, Appl. Math. Lett., 24 (2011) 1–4.
[25] S. Radenović and Z. Kadelburg, Generalized weak contractions in partially ordered metric spaces, Comput. Math. Appl., 60 (2010) 1776-1783.
[26] S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, Some results on weakly contractive maps, Bull. Iranian Math. Soc., 38 (2012) 625–645.
[27] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132 (2004) 1435–1443.
[28] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., 47 (200l) 2683–2693.
[29] M. Turinici, Abstract comparison principles and multivariable Gronwall- Bellman inequalities, J. Math. Anal. Appl., 117 (1986) 100–127.]

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