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Mathematical Modeling of Prey Predator Model with Prey Allee Effect and Predator Harvesting

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Abstract In population dynamics, we propose Allee effect growth Phenomena rate model considered. In this paper, we study real line and interval valued function. Real value function is asymptotically stable, unstable and saddle point exists. Another one model interval value at the point [0,1]. Again at the point p=0, 0.2, 0.5, 0.8, 1 exists. Theoretical analysis and numerical simulation validate the well–know conclusion.

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1. Introduction

In the nonlinear dynamics of the prey predator model differential equation. The study of the dynamics of prey predator systems is one of the dominant subjects in mathematical ecology due to phenomena rate [1, 2, 4]. Considered limited growth prey predator model and Allee effect growth rate due to phenomena function [10]. Thus prey predator models have been in the focus of ecological science from the early days of this discipline [7–9]. It has been turned out very soon that prey predator systems can show different dynamical behaviors such as equilibrium point, stability analysis and phase portrait of system depending on the model parameters [3]. The dynamical analysis of the prey predator model plays an important role in mathematical biology. Though many biologists believe that unique positive equilibrium point of a prey predator system is asymptotically stable, then its limit cycle. Another one model interval value at the interval [0,1] exists, at the point $p = 0, 0.2, 0.5, 0.8, 1$ are asymptotically stable and limit cycle [5, 6]. A prototypical predator prey model considering Allee growth function due to phenomena function and harvesting predator function [3, 6]. Let $x(t)$ and $y(t)$ stands for population

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densities of prey and predator at time t. As follows this equation is nonlinear dynamics of predator prey model.

$$\frac{dx}{dt} = rx(x-a)(b-x) - C_1xy \tag{1.1}$$

$$\frac{dy}{dt} = C_2xy - dy - Ehy \tag{1.2}$$

Where the constant $r, a, b, C_1, C_2, d, E, h$ represent the intrinsic growth rate phenomena, Carrying Capacity of the resource, the threshold value below with the growth rate of the resource is negative. Effect of the density of the predator population due to phenomena on the population growth of the prey. Number of prey necessary to support and replace each individual predator. Intrinsic death rate of predator population. The effort used for harvesting, The catch ability coefficient used for harvesting. Modeling a Strong Allee Effect (SAE) implies $b \le 2$ where as a Weak Allee Effect (WAE) requires $0 \ge b$.

2. Model I Real line function

2.1. Stability and equilibrium point

In this section, we deal with the local stability of the systems (1.1)-(1.2). Let us consider $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$. The critical point are as follows we gives
The prey predator free equilibria $E_0 = \{x = 0, y = 0\}$
The predator free equilibria $E_1 = \{x = b, y = 0\}$
The predator free equilibria $E_2 = \{x = a, y = 0\}$
The coexistence equilibria

$$E_3 = \left\{ \frac{Eh+d}{C_2}, -\frac{r\left(E^2h^2 - EahC_2 - EbhC_2 + abC_2{}^2 + 2\,Edh - adC_2 - bdC_2 + d^2\right)}{C_2{}^2C_1} \right\}$$

The Jacobian matrix of systems (1.1)-(1.2) at the equilibrium point (x,y) is

$$J(x,y) = \left[\begin{array}{cc} r\,(x-a)\,(b-x) + rx\,(b-x) - rx\,(x-a) - C_1y & -C_1x \\ yC_2 & -Eh + xC_2 - d \end{array} \right]$$

2.2. Nature of the equilibrium point and prey predator free equilibria

The first equilibrium point E_0 are the Jacobian evaluated given by

$$J_0 = \left[\begin{array}{cc} -abr & 0 \\ 0 & -Eh - d \end{array} \right]$$

Theorem 2.1. *The equilibrium point (0, 0) exists $\iff$ as follows conditions*

(1) $-abr > 0$ and $-Eh - d > 0$. Then the equilibrium point (0, 0) is unstable.
(2) If $-abr < 0$ and $-Eh - d < 0$. Then the equilibrium point (0, 0) is stable.
(3) If $-abr < 0$ and $-Eh - d < 0$. Then the equilibrium point (0, 0) is saddle point.

Example 2.2. Derive the local stability criterion (1.1)-(1.2) for the model equations.

The equilibrium point (0, 0) is the solution of the equations. The Jacobian matrix of the system evaluated at an equilibrium point (0, 0) is given by $a_{11} = -abr, a_{12} = 0, a_{21} = 0, a_{22} = -Eh - d$

The eigenvalues of J_0 are the roots of $\Lambda_1 = A_2\lambda^2 + A_1\lambda + A_0 = (abr + \lambda)(Eh + d + \lambda)$
By the Routh −Hurwitz theorem conditions for local stability are given as
$A_2 = 1 > 0, A_1 = abr + Eh + d > 0, A_0 = abr(Eh + d) > 0$.

2.3. I. Axial equilibrium point and predator free equilibria

The second equilibrium point $E_1 = \{x = b, y = 0\}$ are Jacobian matrix given by

$$J_1 = \begin{bmatrix} -rb(b-a) & -C_1 b \\ 0 & -Eh + bC_2 - d \end{bmatrix}$$

Theorem 2.3. *The equilibrium point (b, 0) exists* $\iff$ *as follows conditions*

(1) If $abr - b^2r > 0$ *and* $-Eh + bC_2 - d > 0$ *(or) Since* $a < b$ *and* $C_2 < \frac{Eh+d}{b}$. *Then the equilibrium point (b,0) is unstable.*
(2) If $a > b$ *and* $C_2 > \frac{Eh+d}{b}$. *Then the equilibrium point (b,0) is stable*
(3) If $a < b$ *and* $C_2 > \frac{Eh+d}{b}$. *Then the equilibrium point (b,0) is saddle point.*

Example 2.4. Derive the local stability criterion (1.1)-(1.2) for the model equation.

The equilibrium point (b, 0) is the solution of the equations. The Jacobian matrix of the system evaluated at an equilibrium point (b, 0) is given by $a_{11} = -rb(b-a), a_{12} = -C_1 b, a_{21} = 0, a_{22} = -Eh + bC_2 - d$
The eigenvalues of J_1 are $\Lambda_1 = A_2\lambda^2 + A_1\lambda + A_0 = (\lambda - ra(b-a))(Eh - aC_2 + d + \lambda)$
By the Routh−Hurwitz theorem conditions for local stability are given as $A_2 = 1 > 0, A_1 = -abr + b^2r + Eh - bC_2 + d > 0, A_0 = -rb(-b+a)(Eh - bC_2 + d)$
The first condition givens $\frac{Eh+d}{a} > (C_2 - r(a-b))$ (or) $\frac{Eh+d}{a} > C_2$ provided $a > b$.

2.4. II. Axial equilibrium point and predator free equilibria

The third equilibrium point $E_2 = (x = a, y = 0)$ is given by the Jacobian matrix and eigenvalue as

$$J_2 = \begin{bmatrix} ra(b-a) & -C_1 a \\ 0 & -Eh + aC_2 - d \end{bmatrix}$$

$\{-a^2r + abr, -Eh + aC_2 - d\}$

Theorem 2.5. *The equilibrium point (a, 0) exists* $\iff$ *as follows conditions*

(1) If $abr - a^2r > 0$ *and* $-Eh + aC_2 - d > 0$ *(or) Since* $b < a$ *and* $C_2 > \frac{Eh+d}{a}$. *Then the equilibrium point (a, 0) is unstable.*
(2) If $b > a$ *and* $C_2 < \frac{Eh+d}{a}$. *Then the equilibrium point (a, 0) is stable*
(3) If $b < a$ *and* $C_2 > \frac{Eh+d}{a}$. *Then the equilibrium point (a, 0) is saddle point.*

Example 2.6. Derive the local stability criterion (1.1) -(1.2) for the model equation.

The equilibrium point (a, 0) is the solution of the equations. The Jacobian matrix of the system evaluated at an equilibrium point (a, 0) is given by $a_{11} = ra(b-a), a_{12} = -C_1 a, a_{21} = 0, a_{22} = -Eh + aC_2 - d$
The eigenvalues of J_2 are the roots of $\Lambda_2 = A_2\lambda^2 + A_1\lambda + A_0 = (\lambda - ra(b-a))(Eh - aC_2 + d + \lambda)$
By the Routh–Hurwitz theorem conditions for local stability are given as $A_2 = 1 > 0, A_1 = a^2r - abr + Eh - aC_2 + d > 0, A_0 = ra(-b+a)(Eh - aC_2 + d)$
The first condition givens $\frac{Eh - aC_2 + d}{ar} > (a-b)$ (or) $\frac{Eh+d}{a} > C_2$ provided $a > b$.

2.5. Interior equilibrium point and the coexistence equilibria

Example 2.7. Derive the local stability criterion (1.1)-(1.2) for the model equation.

The equilibrium point (x^*, y^*) is the solution of the equations. The Jacobian matrix of the system evaluated at an equilibrium point (x^*, y^*) is given by $a_{11} = \left(\frac{Eh+d}{C_2} - a\right)\left(b - \frac{Eh+d}{C_2}\right) + \frac{(Eh+d)r}{C_2}\left(b - \frac{Eh+d}{C_2}\right) - \frac{(Eh+d)r}{C_2}\left(\frac{Eh+d}{C_2} - a\right) + \frac{r\left(E^2h^2 - EahC_2 - EbhC_2 + abC_2{}^2 + 2\,Edh - adC_2 - bdC_2 + d^2\right)}{C_2{}^2}$,
$a_{12} = -\frac{C_1(Eh+d)}{C_2}$, $a_{11} = -\frac{r\left(E^2h^2 - EahC_2 - EbhC_2 + abC_2{}^2 + 2\,Edh - adC_2 - bdC_2 + d^2\right)}{C_2C_1}$, $a_{22} = 0$.
The eigenvalues of J_3 are the roots of $\Lambda_3 = A_2\lambda^2 + A_1\lambda + A_0 = \frac{-1}{C_2^2}(E^3h^3r - E^2ah^2rC_2 - E^2bh^2rC_2 + EabhrC_2{}^2 + 3\,E^2dh^2r - 2\,E^2h^2\lambda r - 2\,EadhrC_2 + Eah\lambda rC_2 - 2\,EbdhrC_2 + Ebh\lambda rC_2 + abdrC_2{}^2 + 3\,Ed^2hr - 4\,Edh\lambda r - ad^2rC_2 + ad\lambda rC_2 - bd^2rC_2 + bd\lambda rC_2 + d^3r - 2\,d^2\lambda r - \lambda^2C_2{}^2)$
By the Routh−Hurwitz theorem conditions for local stability are given as $A_2 = 1 > 0$, $A_1 = \frac{r(Eh+d)(2\,Eh - aC_2 - bC_2 + 2\,d)}{C_2{}^2} > 0$, $A_0 = -\frac{r(Eh+d)(Eh - bC_2 + d)(Eh - aC_2 + d)}{C_2{}^2} > 0$
The first condition givens $\frac{Eh+d}{(a+b)} > \frac{C_2}{2}$ and $\frac{Eh+d}{b} > C_2 < \frac{Eh+d}{a}$.

3. Model II Interval valued function

An interval number A is represented by closed interval $[a_l, a_r]$ and defined by $A = \{x : a_l \leq x \leq a_r, x \in \mathbb{R}\}$. Where $\mathbb{R}$ is the set of real numbers and a_l, a_r are the left and right limit of the interval number respectively. Also every real number can be represented by the interval number $[a, a]$, for all $a \in \mathbb{R}$.

Definition 3.1. An interval valued number $\widehat{a}$ on [0, 1] is a closed subinterval of [0, 1], that is $\widehat{a} = [a^-, a^+]$ such that $0 \leq a^- \leq a^+ \leq 1$, where a^- and a^+ are the lower and upper limits of $\widehat{a}$ respectively. In this notation $\widehat{0} = [0, 0]$ and $\widehat{1} = [1, 1]$. For any two interval numbers $\widehat{a} = [a^-, a^+]$ and $\widehat{b} = [b^-, b^+]$ on $[0, 1]$ we define,

$$\widehat{a} \leq \widehat{b} \Leftrightarrow a^- \leq b^- and\ a^+ \leq b^+$$

$$\widehat{a} = \widehat{b} \Leftrightarrow a^- = b^- and\ a^+ = b^+$$

Definition 3.2. (Interval valued function) Let $a > 0, b > 0$ and consider the interval $[a, b]$. From a mathematical point of view, any real number can be represented on a line. Similarly we can represent an internal by a function. If the interval is of the form $[a, b]$ the interval valued function is taken as $h(p) = a^{(1-p)}b^p$ for $p \in [0, 1]$

3.1. Parametric interval coefficient and interval coefficient of prey−predator model

Let $\hat{r}, \hat{a}, \hat{b}, \hat{C_1}, \hat{d}, \hat{C_2}$ be the interval counterparts of r, a, b, C_1, C_2, d respectively. The prey predator model with combined harvesting efforts E we get

$$\frac{dx}{dt} = \widehat{r}x(x - \widehat{a})(\widehat{b} - x) - \widehat{C_1}xy \tag{3.1}$$

$$\frac{dy}{dt} = \widehat{C_2}xy - \widehat{d}y - Ehy \tag{3.2}$$

where $\widehat{r} \in [r_l, r_u], \widehat{a} \in [a_l, a_u], \widehat{b} \in [b_l, b_u], \widehat{C_1} \in [C_{1l}, C_{1u}], \widehat{d} \in [d_l, d_u], \widehat{C} \in [C_{2l}, C_{2u}]$. Also $r_l > 0, a_l > 0, C_{1l} > 0, d_l > 0, C_{2l} > 0, b_l > 0$ prey predator model with parametric. Equations (3.1)-(3.2) can be written in the parametric from as follows

$$\frac{dx(t;p)}{dt} = (r_l)^{1-p}(r_u)^p x(x-(a_u)^{1-p}(a_l)^p)((b_l)^{1-p}(b_u)^p - x) - (C_{1u})^{1-p}(C_{1l})^p xy \tag{3.3}$$

$$\frac{dy(t;p)}{dt} = (C_{2l})^{1-p}(C_{2u})^p xy - (d_u)^{1-p}(d_l)^p y - Ehy \tag{3.4}$$

For $p \in [0,1]$. The following theorem gives the parametric form for model (3.3)-(3.4) is possible when $\hat{r}, \hat{a}, \hat{b}, \hat{C}_1, \hat{d}, \hat{C}_2$ are interval numbers.

Theorem 3.3. *The following differential equations with interval valued coefficient*

$$\frac{dx}{dt} = \widehat{r_0}x(x-\widehat{a_0})(\widehat{b_0}-x) - \widehat{C_{10}}xy \tag{3.5}$$

$$\frac{dy}{dt} = \widehat{C_{20}}xy - \widehat{d_0}y - Ehy \tag{3.6}$$

where $\widehat{r_0} \in [r_l, r_u], \widehat{a_0} \in [a_l, a_u], \widehat{b_0} \in [b_l, b_u], \widehat{C_{10}} \in [C_{1l}, C_{1u}], \widehat{d_0} \in [d_l, d_u], \widehat{C_{20}} \in [C_{2l}, C_{2u}]$.
Also $r_l > 0, a_l > 0, b_l > 0, C_{1l} > 0, d_l > 0, C_{2l} > 0$ are provided interval valued functional considered coefficient by the differential equations.

$$\frac{dx(t;p)}{dt} = (r_l)^{1-p}(r_u)^p x(x-(a_u)^{1-p}(a_l)^p)((b_l)^{1-p}(b_u)^p - x) - (C_{1u})^{1-p}(C_{1l})^p xy \tag{3.7}$$

$$\frac{dy(t;p)}{dt} = (C_{2l})^{1-p}(C_{2u})^p xy - (d_u)^{1-p}(d_l)^p y - Ehy \tag{3.8}$$

for $p \in [0,1]$.

Proof. The differential equations (3.7)-(3.8) we get

$$\frac{dx}{dt} = [r_l, r_u]x(x-[a_l, a_u])([b_l, b_u]-x) - [C_{1l}, C_{1u}]x \tag{3.9}$$

$$\frac{dy}{dt} = [C_{2l}, C_{2u}]xy - [d_l, d_u]y - Ehy \tag{3.10}$$

let us consider the parametric values $\widehat{r_1} \in [r_l, r_u], \widehat{a_1} \in [a_l, a_u], \widehat{b_1} \in [b_l, b_u], \widehat{C_{11}} \in [C_{1l}, C_{1u}], \widehat{d_1} \in [d_l, d_u], \widehat{C_{21}} \in [C_{2l}, C_{2u}]$ respectively. Following the interval arithmetic operation and properties of the equations (3.9)-(3.10) reduces to, we get

$$\frac{dx}{dt} = r_1x(x-a_1)(b_1-x) - C_{11}xy \tag{3.11}$$

$$\frac{dy}{dt} = C_{21}xy - d_1y - Ehy \tag{3.12}$$

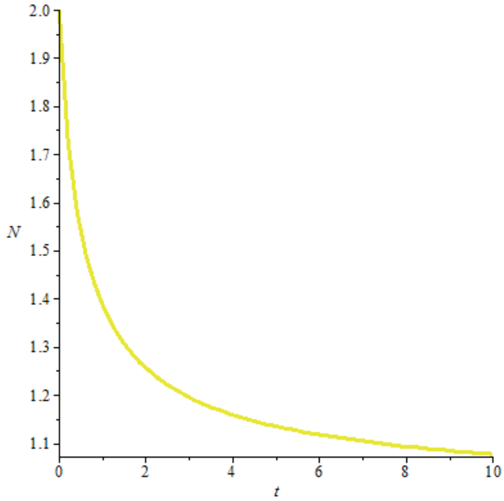

FIGURE 1. Allee growth population at time t.

For fixed m, let us consider the interval valued function $h_m = a_m^{1-p} b_m^p$ for $p \in [0,1]$ an interval $\alpha_m \in [a_m, b_m]$. Since $h_m(p)$ is a strictly increasing and continuous function, the above equation reduces to

$$\frac{dx}{dt} = r_1^{'} x(x - a_1^{'})(b_1^{'} - x)y - C_{11}^{'} x \tag{3.13}$$

$$\frac{dy}{dt} = C_{21}^{'} xy - d_1^{'} y - Ehy \tag{3.14}$$

where the parametric $r_1^{'} \in (r_l)^{1-p}(r_u)^p, a_1^{'} \in (a_u)^{1-p}(a_l)^p, b_1^{'} \in (b_l)^{1-p}(b_u)^p, C_{11}^{'} \in (C_{1l})^{1-p}(C_{1u})^p, d_1^{'} \in (d_l)^{1-p}(d_u)^p, C_{21}^{'} \in (C_{2l})^{1-p}(C_u)^p$ and $p \in [0,1]$. Therefore the parametric form of the differential equations (3.7)–(3.8) is given by

$$\frac{dx(t;p)}{dt} = (r_l)^{1-p}(r_u)^p x(x - (a_u)^{1-p}(a_l)^p)((b_l)^{1-p}(b_u)^p - x) - (C_{1u})^{1-p}(C_{1l})^p xy \tag{3.15}$$

$$\frac{dy(t;p)}{dt} = (C_{2l})^{1-p}(C_{2u})^p xy - (d_u)^{1-p}(d_l)^p y - Ehy \tag{3.16}$$

for $p \in [0,1]$.

4. Numerical Example

Example 4.1. Drown the Allee effect of the prey predator model system $x(t) = rx(t)(x(t) - a)(b - x(t))$ assuming the parameter values as $N(0) = 2, r = 1.0, a = 1, b = 1$ the initial value of time series $t = 0...10$.

The population Allee effect of the prey predator model are given in figure 1.

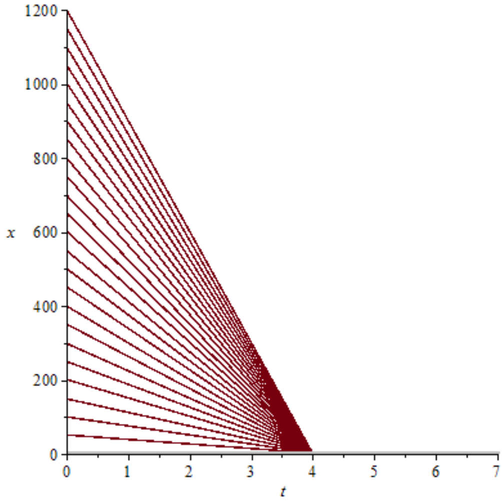

FIGURE 2. Allee growth population at time t.

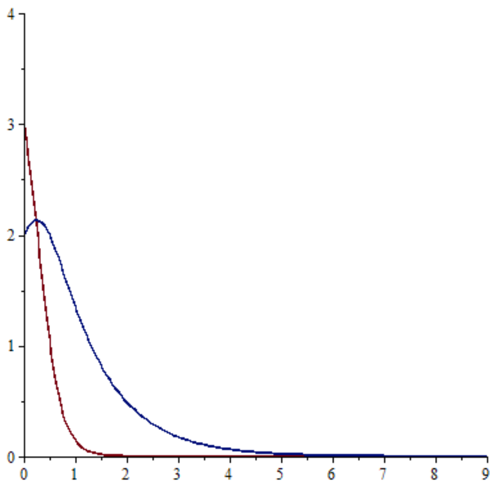

FIGURE 3. Prey predator interaction.

Example 4.2. Drown the time series of the prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0.5, b = 1.0, d = 0.5, C_1, C_2 = 2.0, h = 0.05, E = 0.05$. the initial value $x(0) = 0...100, y(0) = 0...3$.

The time series of the prey model Allee effect given in figure 2.

Example 4.3. Drown the time series of the prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0.5, b = 1.0, d = 0.5, C_1, C_2 = 2.0, h = 0.05, E = 0.05$.

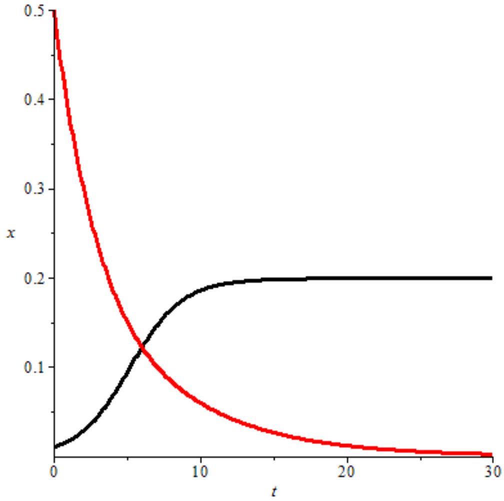

FIGURE 4. Interaction of prey predator population at time t.

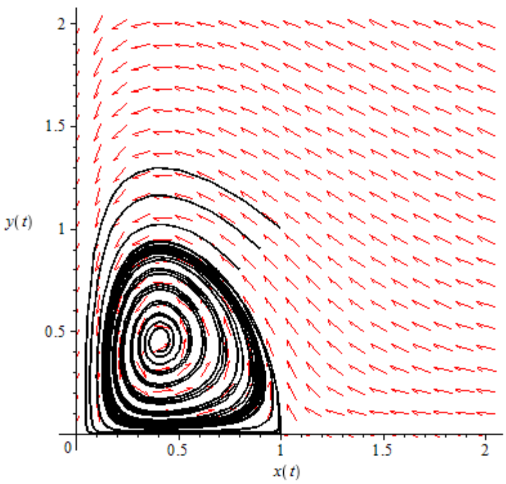

FIGURE 5. Allee effect population of the phase plot at time t.

The interaction of the prey predator model given in figure 3.

Example 4.4. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0, b = 1, C_1 = 1.8, C_2 = 2.5, d = 1, E = 0.01, h = 3$.

Interaction of the prey predator population at time t in figure 4.

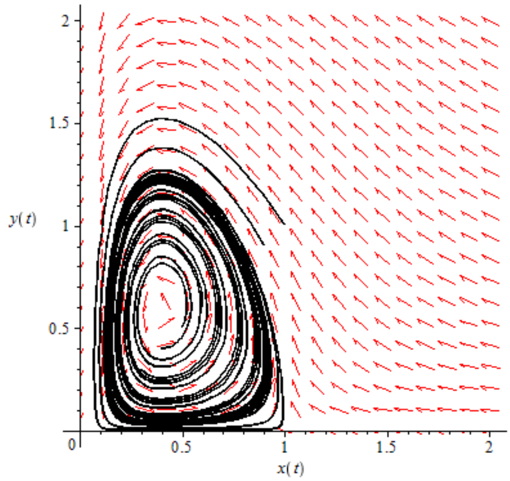

FIGURE 6. Allee effect population of the phase plot at time t.

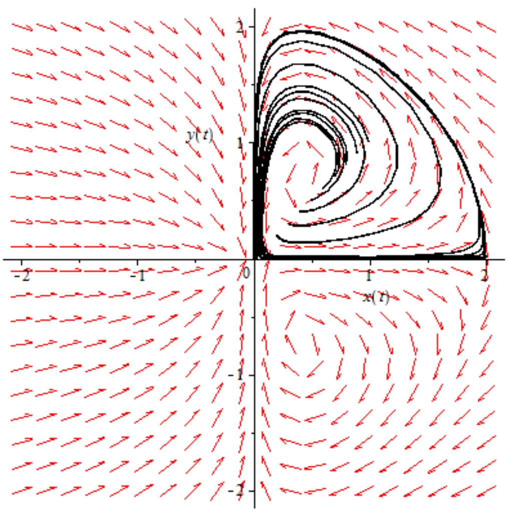

FIGURE 7. Allee effect population of the phase plot at time t.

Example 4.5. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0.002, b = 1, C_1 = 1, C_2 = 2.5, d = 1, E = 0.01, h = 1$.

Allee effect population is asymptotically stable and limit cycle given in figure 5.

Example 4.6. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0, b = 1, C_1 = 1.8, C_2 = 2.5, d = 1, E = 0.01, h = 3$.

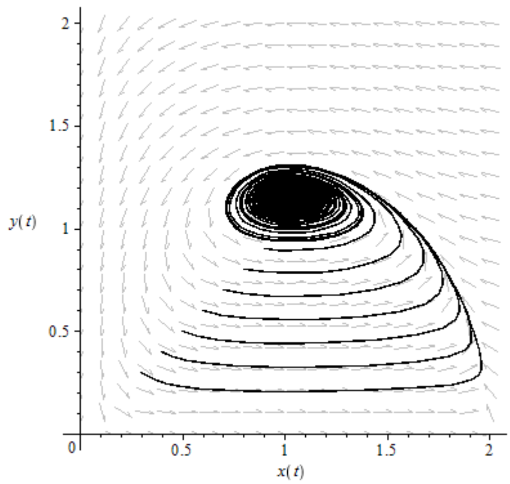

FIGURE 8. Prey predator model is asymptotically stable.

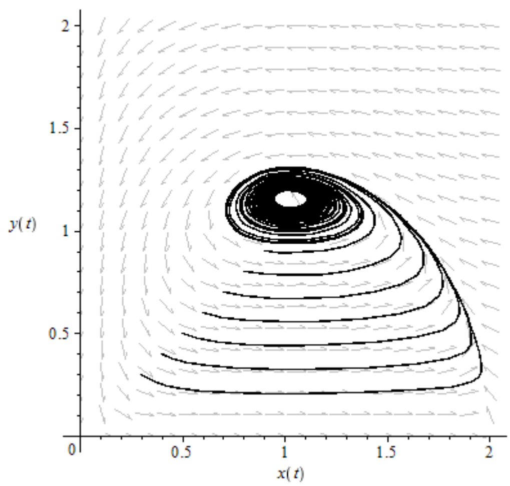

FIGURE 9. Prey predator model is limit cycle.

The phase plot of the prey predator model are Allee effect asymptotically stable and limit cycle given in figure 6.

Example 4.7. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 2.5, a = 0.040, b = 2, C_1 = 1.8, C_2 = 1, d = 1, E = 0.01, h = 3$ at the time series $t = 0...100$.

The phase plot of the prey predator model are asymptotically stable given in figure 7.

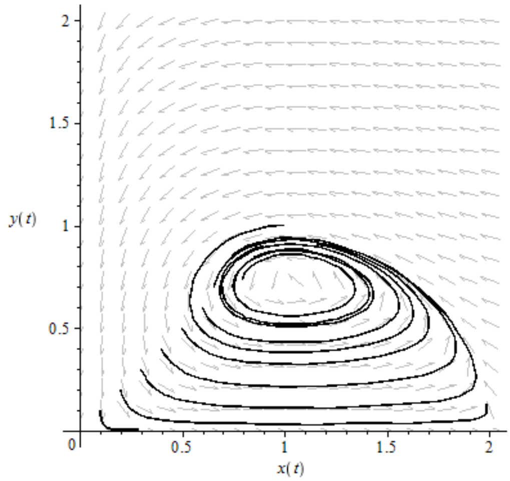

FIGURE 10. Prey predator model is limit cycle.

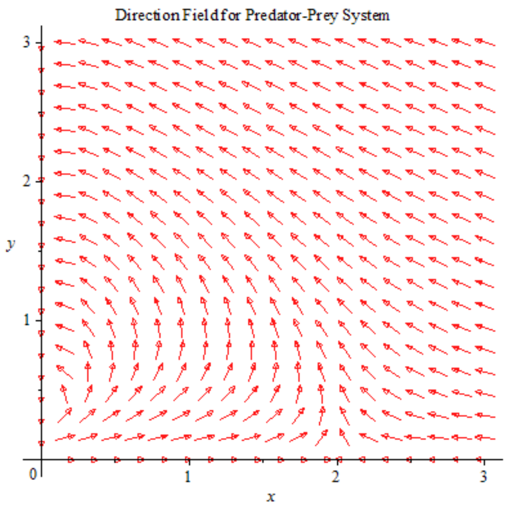

FIGURE 11. Prey predator model is limit cycle.

Example 4.8. Drown nonlinear phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 2.5, a = 0.040, b = 2, C_1 = 1.8, C_2 = 1, d = 1, E = 0.01, h = 3$.

The nonlinear phase plot of the prey predator model are asymptotically stable and limit cycle and saddle point given in figure 8.

Example 4.9. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 2.5, a = 0.040, b = 2, C_1 = 1.8, C_2 = 1, d = 1, E = 0.01, h = 3$.

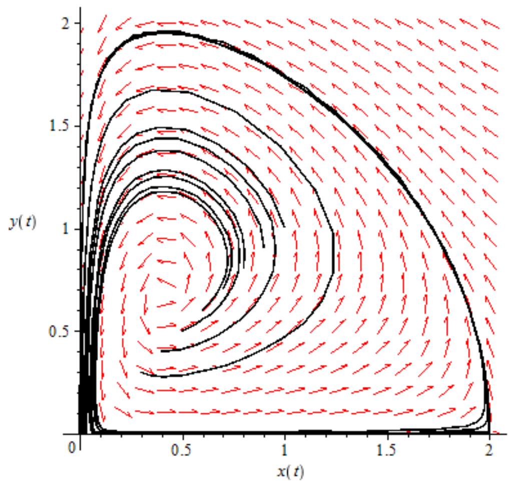

FIGURE 12. Prey predator model is limit cycle.

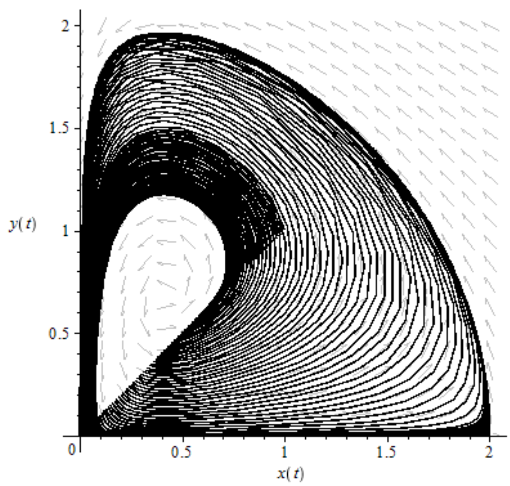

FIGURE 13. Prey predator model is limit cycle.

The phase plot of the prey predator model are asymptotically stable given in figure 9.

Example 4.10. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1, a = 0.040, b = 2, C_1 = 1.8, C_2 = 1, d = 1, E = 0.01, h = 3$.

The phase plot of the prey predator model are asymptotically stable and limit cycle given in figure 10.

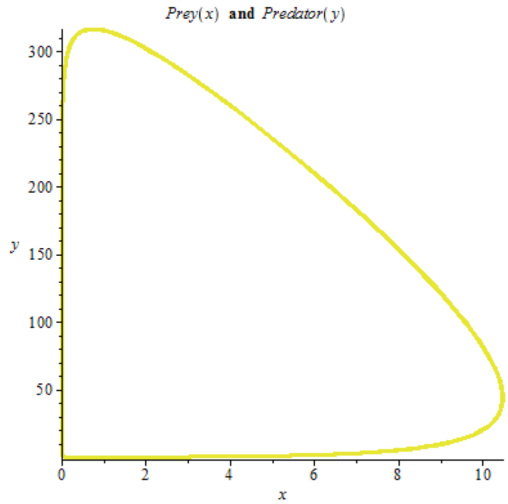

FIGURE 14. Prey predator model in interval [0,1] at the point p=1.

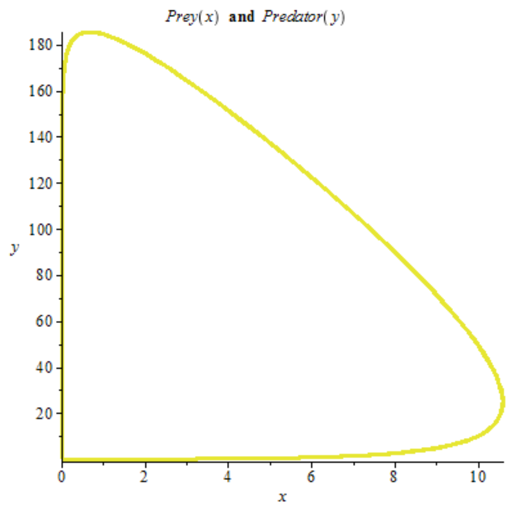

FIGURE 15. Prey predator model in interval [0,1] at the point p=0.8.

Example 4.11. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0.040, b = 2, C_1 = 1.8, C_2 = 2.5, d = 0.01, E = 0.01, h = 3$.

The phase plot of the prey predator model are field prey predator given in figure 11.

Example 4.12. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0, b = 1, C_1 = 1.8, C_2 = 2.5, d = 1, E = 0.1, h = 3$.

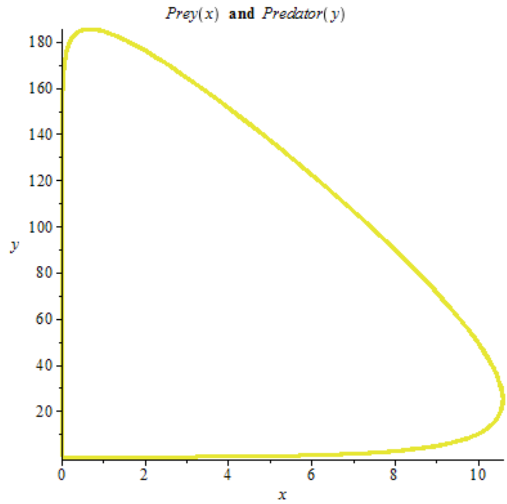

FIGURE 16. Prey predator model in interval [0,1] at the point p=0.5.

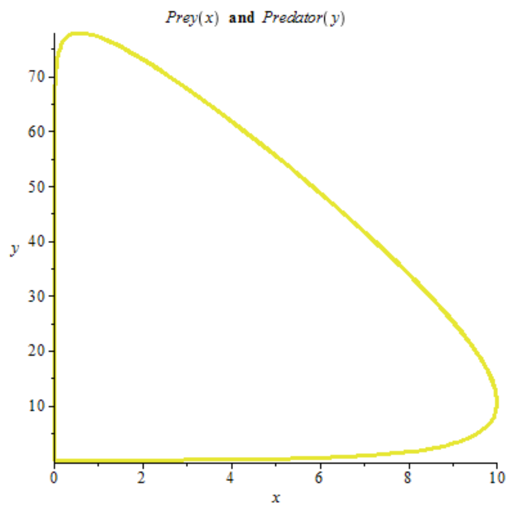

FIGURE 17. Prey predator model in interval [0,1] at the point p=0.2.

Allee effect of prey predator model are asymptotically stable and limit cycle given in figure 12.

Example 4.13. Drown phase plot of the time series of prey predator model system (1.1)-(1.2) assuming the parameter values as $r = 1.5, a = 0.040, b = 2, C_1 = 1.8, C_2 = 2.5, d = 1, E = 0.01, h = 3$.

Allee effect of prey predator model are asymptotically stable and limit cycle given in figure 13.

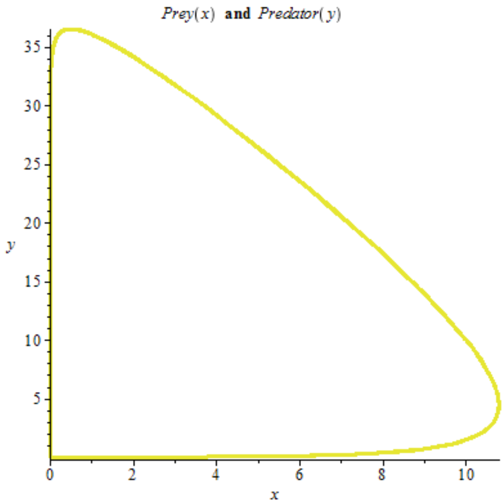

FIGURE 18. Prey predator model in interval [0,1] at the point p=0.

Example 4.14. Drown the interval value at time series of the prey predator model system (3.3)-(3.4) assuming the parameter values as $p = 1, r_l = 2.50, r_u = 2.20, a_l = .15, a_u = .25, b_l = .5, b_u = .4, d_l = .5, d_u = .3, C_{1l} = 1.5, C_{1u} = 1.2, h = 0.4, E = 1.02, C_{2l} = 0.05, C_{2u} = 0.02$ the initial range $t = 0...30$, and initial value (x=10, y=10).

The interval value of the prey predator model are given in figure 14 .

Example 4.15. Drown the interval value at time series of the prey predator model system (3.3)-(3.4) assuming the parameter values as $p = 0.8, r_l = 2.50, r_u = 2.20, a_l = .15, a_u = .25, b_l = .5, b_u = .4, d_l = .5, d_u = .3, C_{1l} = 1.5, C_{1u} = 1.2, h = 0.4, E = 1.02, C_{2l} = 0.05, C_{2u} = 0.02$ the initial range $t = 0...30$, and initial value (x=10,y=10).

The interval value of the prey predator model are given in figure 15.

Example 4.16. Drown the interval value at time series of the prey predator model system (3.3)-(3.4) assuming the parameter values as $p = 0.5, r_l = 2.50, r_u = 2.20, a_l = .15, a_u = .25, b_l = .5, b_u = .4, d_l = .5, d_u = .3, C_{1l} = 1.5, C_{1u} = 1.2, h = 0.4; E = 1.02; C_{2l} = 0.05; C_{2u} = 0.02$ the initial range $t = 0...30$, and initial value (x=10, y=10).

The interval value of the prey predator model are given in figure 16.

Example 4.17. Drown the interval value at time series of the prey predator model systems (3.3)-(3.4) assuming the parameter values as $p = 0.2; r_l = 2.50; r_u = 2.20; a_l = .15; a_u = .25; b_l = .5; b_u = .4; d_l = .5; d_u = .3; C_{1l} = 1.5; C_{1u} = 1.2; h = 0.4; E = 1.02; C_{2l} = 0.05; C_{2u} = 0.02$ the initial range $t = 0...30$, and initial value (x=10, y=10).

The interval value of the prey predator model are given in figure 17.

Example 4.18. Drown the interval value at time series of the prey predator model systems (3.3)-(3.4) assuming the parameter values as $p = 0; r_l = 2.50; r_u = 2.20; a_l = .15; a_u = .25; b_l = .5; b_u = .4; d_l = .5; d_u = .3; C_{1l} = 1.5; C_{1u} = 1.2; h = 0.4; E = 1.02; C_{2l} = 0.05; C_{2u} = 0.02$ the initial range $t = 0...30$, and initial value (x=10, y=10).

The interval value of the prey predator model are given in figure 18.

5. Concussion and result

In this paper, type two model one is real model and another one is interval valued model of Allee effect. Real variable model is Allee effect of asymptotically stable, limit cycle and unstable. Example of Routh–Hurwitz theorem is locally stability analysis exists. Another one model interval valued model at the point $p \in [0, 1]$ is $p = 0, 0.2, 0.5, 0.8, 1$ asymptotically stable and limit cycle exists. Illustrated that the system (3.3)-(3.4) has a cycle trajectory and stability limit cycle.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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