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FIXED POINT RESULTS OF SET VALUED MAPPINGS IN TERMS OF START POINT ON A METRIC SPACE WITH A GRAPH

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Abstract In this article, we define the new concept of start point in a directed graph. We give characterizations for a directed graph to have a start point. The notion of self path set valued map has also been defined and its relation with start point is established in the setting of a metric space with a graph. Further, some fixed point results for set valued maps have been established in this context.

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1. Introduction

One of the most famous fixed point theorem is the Banach Contraction Principle, which initiated a new era of research in fixed point theory due to its immense applicability in major areas of mathematics like numerical analysis and differential/integral equations. This important principle was used by Boyd and Wong [11] to investigate the fixed point results of nonlinear contraction maps.

Study of fixed point results in partially ordered sets has been a very well motivated area of research because of its ease of compatibility in modelling various problems and in finding new convergence schemes. The first attempt in this direction was carried out by Ran and Reurings [23] where he combined the Banach contraction principle and the Knaster-Tarski fixed point theorem. Ran and Reurings considered a class of mappings $f : X \to X$, with (X, d) as a complete metric space and a partial order $\leq$. The mappings they considered were continuous, monotone with respect to the partial order $\leq$. Those

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mappings also satisfy a Banach contraction inequality for every pair $(x, y) \in X \times X$ such that $x \leq y$. When for some $x_0 \in X$, the inequality $x_0 \leq f(x_0)$ is satisfied, they proved that the Picard sequence $\{f^n(x_0)\}$ would converge to a fixed point of f. Ran and Reurings also combined this interesting result with the Schauder fixed point theorem and applied it to obtain some existence and uniqueness results to nonlinear matrix euqations.

Neito & Rodríguez-López ([20], [21]) extended the results of Ran and Reurings to the functions which were not necessarily continuous. The authors also applied their results to obtain a theorem on the existence of a unique solution for periodic boundary problems relative to ordinary differential equations.

Some very important work in this direction that deserve attention are [1, 2, 4–6, 8, 12, 13, 15, 17, 19, 22, 24, 25].

Nadler [18] and Assad and Kirk [7] established some very important fixed point results for set valued and multivalued contraction mappings. Meanwhile, Espinola and Kirk [14] combined the concepts of fixed point theory and graph theory to prove some interesting fixed point theorems in R-trees. In 2008, Jachymski [16] introduced an interesting idea of using the language of graph theory in the study of fixed point results. He was interested in establishing results those would eventually generalize the existing results and also apply the results to the theory of linear operators. So, he studied the class of generalized Banach contractions on a metric space with a directed graph. The advantage of using graph theoretical concepts was that it helped him to describe the results in a unified way and also weaken some conditions significantly. Such works were further extended by Bojor ([9],[10]) in a significant way.

Very recently, some fixed point results on subgraphs of directed graphs were established by Aleomraninejad, Rezapour and Shahzad ([2], [3]). They showed that the Caristi fixed point theorem and a version of Knaster-Tarski fixed point theorem are special cases of their results. Inspired by their work, in the present paper, we prove some fixed point theorems in case of set valued mappings in the setting of a metric space with a graph by defining a new notion called start point of a directed graph.

Let (X, d) be a complete metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, let

$$D(A, B) = \max\{\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)\},$$

where $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(CB(X), D)$ is a metric space and D is said to be a Hausdorff metric induced by d.

Let (X, d) be a metric space and $\Delta = \{(x, x) : x \in X\}$ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set of its vertices coincides with X (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $\Delta \subseteq E(G)$, i.e., G contains all the loops. We assume G has no parallel edges and thus we identify G with the pair $(V(G), E(G))$.

If $x, y \in V(G)$, then a path in G from x to y is a sequence $\{x_i\}_{i=0}^n$ of vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$.

We assume that a path may be finite or infinite and a cycle may be considered as a finite path. Also we assume that G has no isolated vertex.

2. Main Results

Now we are ready to discuss our main results. The following definitions will be useful in this context.

Definition 2.1. Let $CB(X)$ be the class of all non empty closed and bounded subsets of X. For each $x \in V(G)$, the notation $[Tx]_G$ denotes a class of non empty closed and bounded subsets of G such that $[Tx]_G = \{A \in CB(X)$: there exists a path from u to x for some $u \in A\}$.

Definition 2.2. We say that the set valued map $T : X \to CB(X)$ is a self-path map, whenever, for each $x \in V(G)$, there is a path from u to x for some $u \in Tx$, we denote this by $Tx \in [Tx]_G$.

When $x \neq y$, by the notation $Tx \in [Ty]_G$ we mean that there is a path from x to Ty for some $u \in Ty$.

Also, we define $[y]_{G^l}$ as $[y]_{G^l} = \{x \in G :$ there exists a path from y to $x\}$.

Furthermore, the point $x \in V(G)$ is said to be a fixed point of the set valued map $T : X \to CB(X)$ if $x \in Tx$.

The next example motivates the study of multivalued mappings by showing that control problems may be translated in terms of multivalued maps, and hence, study of their fixed points could provide new solution schemes to such problems.

Example 2.3. Suppose that the following control problem is to be solved:

$$x^{'}(t) = f(t, x(t), u(t)),$$
$$\text{and } x(0) = x_0,$$

which is controlled by parameters $u(t)$ (called the controls), where $f : [0, a] \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

To solve the above problem we define a multivalued map $F : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$F(t, x) = \{f(t, x, u)\}_{u \in U}.$$

Then solutions of the above problem are solutions of the following differential inclusions:

$$x^{'}(t) \in F(t, x(t)),$$
$$\text{and } x(0) = x_0.$$

Definition 2.4. Let $G^{'}$ be a subgraph of the directed graph G. We say that $c \in V(G)$ is a lower bound for $G^{'}$ whenever $g^{'} \in [c]_{G^l}$ for all $g^{'} \in V(G^{'})$. Also, we say that $d \in V(G)$ is an infimum of $G^{'}$ whenever $d \in [c]_{G^l}$ for all lower bounds c.

Definition 2.5. Let G be a directed graph and $u \in V(G)$. We say that u is a start point whenever there is no $x \in V(G)$ such that $x \neq u$ and $(x, u) \in E(G)$.

Definition 2.6. Let H be a subgraph of the directed graph G. A vertex $s \in V(H)$ is said to be a start point of H if there is no $x \in V(H)$ such that $x \neq s$ and $(x, s) \in E(G)$.

Definition 2.7. Let (X, d) be a metric space and $\phi : X \to (-\infty, \infty)$ a map. Suppose that G is the directed graph defined by $V(G) = X$ and $E(G) = \{(x, y) : d(x, y) \leq \phi(x) - \phi(y)\}$. We say that ϕ is upper semi continuous whenever $\phi(x_n) \leq \phi(x)$ for all sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to x.

Our first result uses the concept of a minimal path. If Z denotes the set of all paths in a directed graph G, then $(Z, \subseteq)$ is a partially ordered set. Also since it is trivially true that every partially ordered set has a minimal element, we can conclude that G has a minimal path.

Theorem 2.8. *Let G be a directed graph such that every path in G has a lower bound within itself. Then there exists a path in G, considered as a subgraph of G, which has a start point or a cycle.*

Proof. Let us assume that G has no cycle. Let B be a minimal path in G and $l \in V(B)$ be a lower bound of B. If l is not a start point, there exists $x \in V(B)$ such that $x \neq l$ and $(x, l) \in E(G)$. Then $B \setminus \{x\}$ is a path in G and $B \setminus \{x\} \subset B$. This contradicts the fact that B is a minimal path. Hence l must be a start point of B. ∎

Theorem 2.9. *Let G be a directed graph. Then G has a start point if and only if each self path map on G has a fixed point.*

Proof. Let G has a start point e and T be a self path map on G. We claim that e is a fixed point of T. Since $Te \in [Te]_G$, there is a path (finite or infinite) from u to e for some $u \in Te$. But since e is a start point, this is not possible for any e unless $u = e$. Thus, we must have $e \in Te$, i.e., e is a fixed point of T.

Again, suppose G is a directed graph and each self path map on G has a fixed point. If possible assume that G has no start point. Then for each $x \in V(G)$, there exists $y \in V(G)$ such that $y \neq x$ and $(y, x) \in E(G)$. Now, considering every such pair $(y, x) \in E(G)$, we can define a map $T : G \to CB(G)$ such that for each $x \in V(G)$, $Tx = \{y\}$. It is easy to see that T is a self path map because $Tx = \{y\} \in [Tx]_G$, for all $x \in V(G)$, but T has no fixed point as $x \notin Tx$ for any $x \in V(G)$. This contradicts our hypothesis and thus G has a start point. ∎

Our next example shows that indeed if a directed graph has no start point, then a self-path map may be defined which has no fixed point.

Example 2.10. Let G be a directed graph with vertices $V(G) = \{a, b, c, d, e\}$ and the edges $E(G) = \{(e, d), (d, c), (c, b)\} \bigcup \{(a, e), (a, d), (c, a), (b, a)\}$. Define the map $T : V(G) \to CB(X)$ such that $Ta = \{b\}, Tb = \{c\}, Tc = \{d\}, Td = \{e\}, Te = \{a\}$. Then it is easy to see that T is a self path map but G has no fixed point and no start point either.

Theorem 2.11. *Let G be a directed graph such that every path in G has an infimum within itself and $T : V(G) \to CB(X)$ be a self-path map. Also let $G^{'} = \{x \in V(G) : Tx \in [Tx]_G\}$ and $G^{'}$ has no cycle. Then T has a fixed point in $G^{'}$.*

Proof. Let B be a path in $G^{'}$ and $b \in V(G)$ be its infimum (greatest lower bound). Therefore, from the definition of self path map, we have $Tb \in [Tb]_G$, which implies that $b \in V(G^{'})$. Also, $G^{'}$ is a subgraph of G. Now, using Theorem 2.8, $G^{'}$ has a start point. Again, since T may be considered as a self-path map on G', using Theorem 2.9 we can conclude that T has a fixed point in $G^{'}$. ∎

Lemma 2.12. *Let X be a complete metric space and $\phi : X \to (-\infty, \infty)$ a map bounded from above. Suppose that G is the directed graph defined by $V(G) = X$ and $E(G) = \{(x, y) : d(x, y) \leq \phi(x) - \phi(y)\}$. If ϕ is upper semi continuous, then G has a start point.*

Proof. First we prove that G has no cycle. If G has a cycle, then there exist a finite path $\{\lambda_i\}_{i=1}^n$ in G such that $\lambda_1 = \lambda_n$ (for, in a cycle, initial and terminal vertices are same).

Now, $d(\lambda_i, \lambda_1) \leq \phi(\lambda_i) - \phi(\lambda_1)$, and also, $d(\lambda_i, \lambda_1) \leq \phi(\lambda_1) - \phi(\lambda_i) = -[\phi(\lambda_i) - \phi(\lambda_1)]$. But this is possible only when $\phi(\lambda_i) = \phi(\lambda_1)$, i.e., $d(\lambda_i, \lambda_1) \leq 0$. This implies that $\lambda_i = \lambda_1$ for all $i \geq 2$, which is a contradiction. Thus, G can not have any cycle.

Next we show that each path in G has a lower bound. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a path in G. Then $\{\phi(x_\lambda)\}_{\lambda\in\Lambda}$ is an increasing net of real numbers. As ϕ is bounded from above, we can obtain a decreasing sequence $\{\lambda_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} \phi(x_{\lambda_n}) = \sup_{\lambda\in\Lambda} \phi(x_\lambda)$.

Now,

$$\begin{aligned}
d(x_{\lambda_n}, x_{\lambda_m}) &\leq d(x_{\lambda_n}, x) + d(x, x_{\lambda_m}), \text{ for } n, m \geq 1 \\
&= \phi(x_{\lambda_n}) - \phi(x) + \phi(x) - \phi(x_{\lambda_m}) \\
&= \phi(x_{\lambda_n}) - \phi(x_{\lambda_m}) \\
\Rightarrow \lim_{n,m\to\infty} d(x_{\lambda_n}, x_{\lambda_m}) &\leq \lim_{n\to\infty} \phi(x_{\lambda_n}) - \lim_{m\to\infty} \phi(x_{\lambda_m}) \\
&= \sup \phi(x_\lambda) - \sup \phi(x_\lambda) \\
&= 0.
\end{aligned}$$

Therefore, we have that $\{x_{\lambda_n}\}_{n\geq 1}$ is a Cauchy sequence. Since X is complete, $\{x_{\lambda_n}\}$ must converge to some $x \in X$. As ϕ is upper semi continuous, we now have $\phi(x_{\lambda_n}) \leq \phi(x)$ $\Rightarrow 0 \leq \phi(x) - \phi(x_{\lambda_n})$ i.e., $d(x, x_{\lambda_n}) \leq \phi(x) - \phi(x_{\lambda_n})$.

Thus, $x_{\lambda_n} \in [x]_{G^l}$ for all $n \geq 1$. So, x is a lower bound for $\{x_{\lambda_n}\}_{n\geq 1}$. Now we show that x is a lower bound for $\{x_\lambda\}_{\lambda\in\Lambda}$. If there exists $\mu \in \Lambda$ such that $x_{\lambda_n} \in [x_\mu]_{G^l}$, for all $n \geq 1$, then $\phi(x_{\lambda_n}) \leq \phi(x_\mu)$ for all $n \geq 1$ which implies that $\phi(x_\mu) = \sup_{\lambda\in\Lambda} \phi(x_\lambda)$. Since $d(x_{\lambda_n}, x_\mu) \leq \phi(x_{\lambda_n}) - \phi(x_\mu)$, from the definition of upper semi continuous map, we have $x_{\lambda_n} \to x_\mu$. This implies that $x_\mu = x$ (for $x_{\lambda_n} \to x$). Hence $\phi(x) = \sup_{\lambda\in\Lambda} \phi(x_\lambda)$. We claim that $x_\lambda \in [x]_{G^l}$. In fact, if there is $\lambda \in \Lambda$ such that $x \in [x_\lambda]_{G^l}$ then $d(x_\lambda, x) \leq \phi(x_\lambda) - \phi(x) \leq \phi(x_\lambda) - \phi(x_\lambda) = 0$, and so, $x_\lambda = x$. Since $\{x_\lambda\}_{\lambda\in\Lambda}$ is a path in G, if the previous case is not true, then for each $\lambda \in \Lambda$, there exists $n \geq 1$ such that $x_\lambda \in [x_{\lambda_n}]_{G^l}$.

Again we have $x_{\lambda_n} \in [x]_{G^l}$. This implies that $x_\lambda \in [x]_{G^l}$. Thus x is a lower bound for $\{x_\lambda\}_{\lambda\in\Lambda}$. Now, using Theorem 2.8, we can say that G has a start point. ∎

Theorem 2.13. *Let (X, d) be a complete metric space, $\phi : X \to (-\infty, \infty)$ a map bounded from above and upper semi continuous and $T : X \to CB(X)$ a self path map satisfying the condition $d(u, x) \leq \phi(u) - \phi(x)$, for all $x \in X$ and $u \in Tx$. Then T has a fixed point.*

Proof. Suppose that G is the directed graph via the vertices $V(G) = X$ and the edges $E(G) = \{(x, y) : d(x, y) \leq \phi(x) - \phi(y)\}$. Using Lemma 2.12, we can conclude that G has a start point. Again, using Theorem 2.9, it is trivial to see that T has a fixed point. ∎

Conclusion. In this paper we introduced the new concept of start point in a directed graph and established some fixed point results for set valued mappings in terms of start point in the setting of a metric space endowed with a directed graph. The results discussed in this paper are mainly concerned with the existence of fixed points. The study of uniqueness of fixed points in the current context would be an interesting topic for future study.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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