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FIXED POINT THEOREMS IN GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES

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Abstract In this paper, we introduce generalized intuitionistic fuzzy metric spaces and discuss their properties. As an application of this concept, we prove coupled common fixed point theorems for mixed weakly monotone maps in partial ordered in generalized intuitionistic fuzzy metric spaces.

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1. Introduction and Preliminaries

The notion of fuzzy sets was initially investigated by Zadeh [15] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. Attanassov [2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park [10] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and

Veeramani [3]. Mustafa and Sims [7] introduced a G-metric space and obtained some fixed point theorems in it. Abbas et al. [1] established the notion of A- metric spaces, a generalization of S- metric space [12]. Vishal Gupta and Ashima Kanwar [14] introduce the V- fuzzy metric space. We introduce new generalized intuitionistic fuzzy metric spaces and discuss their properties. We prove coupled common fixed point theorems for mixed weakly monotone maps in partial ordered in generalized intuitionistic fuzzy metric spaces.

Definition 1.1. [1] Let X be a nonempty set.A function $A : X^n \to [0, +\infty]$ is called an A-metric on X if for any $x_i, a \in X, I = 1, 2, 3, ..., n$, the following conditions hold:
$(A-1)$ $A(x_1, x_2, x_3, ..., x_n) \geq 0$,
$(A-2)$ $A(x_1, x_2, x_3, ..., x_n) = 0$, if and only if $x_1 = x_2 = x_3 = ... = x_n = 0$,
$(A-3)$ $A(x_1, x_2, x_3, ..., x_n) \leq A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) + A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a)$
$+... + A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a)$.
The pair (X, A) is called an A -metric space.

Example 1.2. [1] Let $X = R$ Define the function $A : X^n \to [0, +\infty]$ by
$A(x_1, x_2, x_3, ..., x_n) = \Sigma_{i=1}^{n} \Sigma_{i>j} |x_i - y_j|$. Then (X, A) is called the usual A-metric space.

Definition 1.3. [14] Let X be a nonempty set. A triple $(X, V, *)$ is said to be a V- fuzzy metric space (denoted by VF -space), where $*$ is a continuous t-norm, and V is a fuzzy set on $X^n \times (0, \infty)$ satisfying the following conditions for all $t, s > 0$:
$(VF-1)$ $V(x, x, x, ..., x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,
$(VF-2)$ $V(x_1, x_1, x_1, ..., x_1, x_2, t) \geq V(x_1, x_2, x_3, ..., x_n, t)$ for all $x_1, x_2, x_3, ..., x_n \in X$ with $x_2 \neq x_3 \neq ... \neq x_n$,
$(VF-3)$ $V(x_1, x_2, x_3, ..., x_n, t) = 1$ if and only if $x_1 = x_2 = x_3 = ... = x_n$,
$(VF-4)$ $V(x_1, x_2, x_3, ..., x_n, t) = V(p(x_1, x_2, x_3, ..., x_n), t)$ where, p is a permutation function,
$(VF-5)$ $V(x_1, x_2, x_3, ..., x_{n-1}, t + s) \geq V(x_1, x_2, x_3, ..., x_{n-1}, l, t) * V(l, l, l, ..., l, x_n, s)$,
$(VF-6)$ $\lim_{n\to\infty} V(x_1, x_2, x_3, ..., x_n, t) = 1$,
$(VF-7)$ $V(x_1, x_2, x_3, ..., x_n, .) : (0, \infty) \to [0, 1]$ is continuous.

2. Generalized Intuitionistic Fuzzy Metric Space

Definition 2.1. Let X be a nonempty set.A triple $(X, V, W, *, \diamondsuit)$ is said to be a generalized intuitionistic fuzzy metric space (denoted by GIFM -space),where $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm and V, W are fuzzy sets on $X^n \times (0, \infty)$ satisfying the following conditions, for every $x_1, x_2, x_3, ..., x_n, l \in X, t, s > 0$,

(i) $V(x_1, x_2, x_3, ..., x_n, t) + W(x_1, x_2, x_3, ..., x_n, t) \leq 1$,
(ii) $V(x, x, x, ..., x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,
(iii) $V(x_1, x_1, x_1, ..., x_1, x_2, t) \geq V(x_1, x_2, x_3, ..., x_n, t)$ for all $x_1, x_2, x_3, ..., x_n \in X$ with $x_2 \neq x_3 \neq ... \neq x_n$,
(iv) $V(x_1, x_2, x_3, ..., x_n, t) = 1$ if and only if $x_1 = x_2 = x_3 = ... = x_n$,
(v) $V(x_1, x_2, x_3, ..., x_n, t) = V(p(x_1, x_2, x_3, ..., x_n), t)$, where p is a permutation function,
(vi) $V(x_1, x_2, x_3, ..., x_{n-1}, t + s) \geq V(x_1, x_2, x_3, ..., x_{n-1}, l, t) * V(l, l, l, ..., l, x_n, s)$,
(vii) $V(x_1, x_2, x_3, ..., x_n, .) : (0, \infty) \to [0, 1]$ is continuous,
$(viii)$ V is a non decreasing function on R^+, $\lim_{n\to\infty} V(x_1, x_2, x_3, ..., x_n, t) = 1$ and $\lim_{t\to 0} V(x_1, x_2, x_3, ..., x_n, t) = 0$, for all $x_1, x_2, x_3, ..., x_n \in X, t > 0$,
(ix) $W(x, x, x, ..., x, y, t) < 1$ for all $x, y \in X$ with $x \neq y$,

(x) $W(x_1, x_1, x_1, ..., x_1, x_2, t) \leq W(x_1, x_2, x_3, ..., x_n, t)$ for all $x_1, x_2, x_3, ..., x_n \in X$ with $x_2 \neq x_3 \neq ... \neq x_n$,

(xi) $W(x_1, x_2, x_3, ..., x_n, t) = 0$ if and only if $x_1 = x_2 = x_3 = ... = x_n$,

(xii) $W(x_1, x_2, x_3, ..., x_n, t) = W(p(x_1, x_2, x_3, ..., x_n), t)$, where p is a permutation function,

$(xiii)$ $W(x_1, x_2, x_3, ..., x_{n-1}, t+s) \leq W(x_1, x_2, x_3, ..., x_{n-1}, l, t) \diamondsuit W(l, l, l, ..., l, x_n, s)$,

(xiv) $W(x_1, x_2, x_3, ..., x_n, .) : (0, \infty) \to [0, 1]$ is continuous,

(xv) W is a non increasing function on R^+, $\lim_{n\to\infty} W(x_1, x_2, x_3, ..., x_n, t) = 0$, and $\lim_{t\to 0} W(x_1, x_2, x_3, ..., x_n, t) = 1$, for all $x_1, x_2, x_3, ..., x_n \in X, t > 0$.

In this case, the pair (V, W) is called an generalized intuitionistic fuzzy metric spaces.

Example 2.2. Let (X, V) be a V - metric space. For all $x_1, x_2, x_3, ..., x_n \in X$ and every $t > 0$, consider (V, W) to be fuzzy sets on $X^n \times (0, \infty)$ define by $V(x_1, x_2, x_3, ..., x_n, t) = \dfrac{t}{t + A(x_1, x_2, x_3, ..., x_n)}$, $W(x_1, x_2, x_3, ..., x_n, t) = \dfrac{A(x_1, x_2, x_3, ..., x_n)}{t + A(x_1, x_2, x_3, ..., x_n)}$ and denote $a * b = ab$ and $a \diamondsuit b = \min\{a+b, 1\}$. Then $(X, V, W, *, \diamondsuit)$ is an generalized intuitionistic fuzzy metric spaces.

Lemma 2.3. *Let $(X, V, W, *, \diamondsuit)$ be a generalized intuitionistic fuzzy metric space. Then $V(x_1, x_2, x_3, ..., x_n, t)$ is non-decreasing and $W(x_1, x_2, x_3, ..., x_n, t)$ non-increasing with respect to t.*

Proof. Since $t > 0$ and $t + s > t$ for $s > 0$, by letting $l = x_n$ is a condition (vi) and (xiii) of a (V, W) are fuzzy metric space, we get
$V(x_1, x_2, x_3, ..., x_{n-1}, x_n, t+s) \geq V(x_1, x_2, x_3, ..., x_{n-1}, x_n, t) * V(x_n, x_n, x_n, ..., x_n, x_n, s)$
and
$W(x_1, x_2, x_3, ..., x_{n-1}, x_n, t+s) \leq W(x_1, x_2, x_3, ..., x_{n-1}, x_n, t) \diamondsuit W(x_n, x_n, x_n, ..., x_n, x_n, s)$
This implies that

$$V(x_1, x_2, x_3, ..., x_{n-1}, x_n, t+s) \geq V(x_1, x_2, x_3, ..., x_{n-1}, x_n, t)$$

and

$$W(x_1, x_2, x_3, ..., x_{n-1}, x_n, t+s) \leq W(x_1, x_2, x_3, ..., x_{n-1}, x_n, t).$$

So, $V(x_1, x_2, x_3, ..., x_n, t)$ is non-decreasing and $W(x_1, x_2, x_3, ..., x_n, t)$ non-increasing with respect to t. ∎

Lemma 2.4. *Let $(X, V, W, *, \diamondsuit)$ be a generalized intuitionistic fuzzy metric space such that*

$$V(x_1, x_2, x_3, ..., x_n, kt) \geq V(x_1, x_2, x_3, ..., x_n, t)$$

and

$$W(x_1, x_2, x_3, ..., x_n, kt) \leq W(x_1, x_2, x_3, ..., x_n, t)$$

with $k \in (0, 1)$. Then $x_1 = x_2 = x_3 = ... = x_n$

Proof. By assumption,

$$\begin{aligned} V(x_1, x_2, x_3, ..., x_n, kt) &\geq V(x_1, x_2, x_3, ..., x_n, t) \text{ and} \\ W(x_1, x_2, x_3, ..., x_n, kt) &\leq W(x_1, x_2, x_3, ..., x_n, t) \text{ for } t > 0. \end{aligned} \tag{2.1}$$

Since $kt < t$ and by Lemma 2.3, we have

$$\begin{aligned} V(x_1, x_2, x_3, ..., x_n, kt) &\leq V(x_1, x_2, x_3, ..., x_n, t) \text{ and} \\ W(x_1, x_2, x_3, ..., x_n, kt) &\geq W(x_1, x_2, x_3, ..., x_n, t). \end{aligned} \tag{2.2}$$

From (2.1), (2.2) and the definition of a generalized intuitionistic fuzzy metric space, we get $x_1 = x_2 = x_3 = ... = x_n$. ∎

Definition 2.5. Let $(X, V, W, *, \diamondsuit)$ be a generalized intuitionistic fuzzy metric space. A sequence $\{x_r\}$ is said to converge to a point $x \in X$ if $V(x_r, x_r, x_r, ..., x_r, x, t) \to 1$ and $W(x_r, x_r, x_r, ..., x_r, x, t) \to 0$ as $r \to \infty$ for all $t > 0$, that is, for each $\epsilon > 0$, there exists $n \in N$ such that for all $r \geq N$, we have

$$V(x_r, x_r, x_r, ..., x_r, x, t) > 1 - \epsilon \text{ and } W(x_r, x_r, x_r, ..., x_r, x, t) < \epsilon,$$

we write $\lim_{r\to\infty} x_r = x$.

Definition 2.6. Let $(X, V, W, *, \diamondsuit)$ be a generalized intuitionistic fuzzy metric space. A sequence $\{x_r\}$ is said to a Cauchy sequence if $V(x_r, x_r, x_r, ..., x_r, x_q, t) \to 1$ and $W(x_r, x_r, x_r, ..., x_r, x_q, t) \to 0$ as $r, q \to \infty$ for all $t > 0$, that is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $r, q \geq n_0$, we have

$$V(x_r, x_r, x_r, ..., x_r, x_q, t) > 1 - \epsilon \text{ and } W(x_r, x_r, x_r, ..., x_r, x_q, t) < \epsilon.$$

Definition 2.7. The (V, W) fuzzy metric space $(X, V, W, *, \diamondsuit)$ is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.8. Let $(X, V, W, *, \diamondsuit)$ be a generalized intuitionistic fuzzy metric space.The mappings P and Q, where $P : X \times X \to X$ and $Q : X \to X$ are said to be compatible with respect to (V, W) if for all $t > 0$,

$$\lim_{r\to\infty} V(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(Q(x_r), Q(y_r)), t) = 1,$$
$$\lim_{r\to\infty} W(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(Q(x_r), Q(y_r)), t) = 0,$$
$$\lim_{r\to\infty} V(Q(P(y_r, x_r)), Q(P(y_r, x_r)), ..., Q(P(y_r, x_r)), P(Q(y_r), Q(x_r)), t) = 1,$$

and

$$\lim_{r\to\infty} W(Q(P(y_r, x_r)), Q(P(y_r, x_r)), ..., Q(P(y_r, x_r)), P(Q(y_r), Q(x_r)), t) = 0,$$

where $\{x_r\}$ and $\{y_r\}$ are sequences in X such that $\lim_{r\to\infty} Q(x_r) = \lim_{r\to\infty} P(x_r, y_r) = x$ and $\lim_{r\to\infty} Q(y_r) = \lim_{r\to\infty} P(y_r, x_r) = y$ for all $x, y \in X$ and $t > 0$.

3. Main Results

In this section, we explicitly prove fixed point theorems for coupled maps on partially ordered generalized intuitionistic fuzzy metric space.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and $(X, V, W, *, \diamondsuit)$ be a complete generalized intuitionistic fuzzy metric space. Suppose that $P : X \times X \to X$ and $Q : X \times X \to X$ are mappings such that*
(3.1.1) $P(X \times X) \subseteq Q(X)$

(3.1.2) *P has the mixed Q - monotone property,*
(3.1.3) *there exist $k \in (0,1)$ such that*

$$\begin{aligned} V(P(x,y),P(x,y),...,P(x,y),P(u,v),kt) \geq\ & \{V(Q_x,Q_x,...,Q_x,Q_u,t) \\ & *V(Q_x,Q_x,...,Q_x,P(x,y),t) \\ & *V(Q_u,Q_u,...,Q_u,P(u,v),t)\} \end{aligned}$$

$$\begin{aligned} W(P(x,y),P(x,y),...,P(x,y),P(u,v),kt) \leq\ & \{W(Q_x,Q_x,...,Q_x,Q_u,t) \\ & \Diamond W(Q_x,Q_x,...,Q_x,P(x,y),t) \\ & \Diamond W(Q_u,Q_u,...,Q_u,P(u,v),t)\} \end{aligned}$$

for all $x,y,u,v \in X$ and $t>0$ for which $Q(x) \leq Q(u)$ and $Q(y) \geq Q(v)$ or $Q(x) \geq Q(u)$ and $Q(y) \leq Q(v)$,
(3.1.4) *Q is continuous, and P and Q are compatible.*
Also suppose that
(a) P is continuous or
(b) X has the following properties:
(i) If $\{x_r\}$ is a non-decreasing sequence such that $x_r \to x$, then$x_r \leq x$ for all $r \in N$,
(ii) If $\{y_r\}$ is a non-decreasing sequence such that $y_r \to y$, then$y_r \geq y$ for all $r \in N$,
If there exist $x_0, y_0 \in X$ such that $Q(x_0) \leq P(x_0,y_0)$ and $Q(y_0) \geq P(y_0,x_0)$, then P and Q have a coupled coincidence point in X.

Proof. Let (x_0,y_0) be a given point in $X \times X$ such that $Q(x_0) \leq P(x_0,y_0)$ and $Q(y_0) \geq P(y_0,x_0)$. Using (3.1.1), choose x_1, y_1 such that

$$P(x_0,y_0) = Q(x_1) \ \text{ and } P(y_0,x_0) = Q(y_1). \tag{3.1}$$

Construct two sequences $\{x_r\}$ and $\{y_r\}$ in X such that

$$P(x_r,y_r) = Q(x_{r+1}) \ \text{ and } \ P(y_r,x_r) = Q(y_{r+1}) \ \text{ for all} r \geq 0. \tag{3.2}$$

Now we shall prove that

$$Q(x_r) \leq Q(x_{r+1}) \ \text{ and } Q(y_r) \geq Q(y_{r+1}) \tag{3.3}$$

We use mathematical induction.

Step 1: Let $r=0$. Since $Q(x_0) \leq P(x_0,y_0)$ and $Q(y_0) \geq P(y_0,x_0)$. Using condition (3.1), we have $Q(x_0) \leq Q(x_1)$ and $Q(y_0) \geq Q(y_1)$. So inequalities (3.3) hold for $r=0$.

Step 2: Now suppose that (3.3) hold for some fixed $s \geq 0$. So we get $Q(x_s) \leq Q(x_{s+1})$ and $Q(y_s) \geq Q(y_{s+1})$.

Step 3: Since P has the mixed Q-monotone property, using (3.1.6) we have

$$Q(x_{r+1}) = P(x_r,y_r) \leq P(x_{r+1},y_r) \ \text{ and } Q(y_{r+1}) = P(y_r,x_r) \geq P(y_{r+1},x_r) \tag{3.4}$$

Also,

$$Q(x_{r+2}) = P(x_{r+1},y_{r+1}) \geq P(x_{r+1},y_r) \ \text{ and } Q(y_{r+2}) = P(y_{r+1},x_{r+1}) \leq P(y_{r+1},x_r) \tag{3.5}$$

From (3.4 and (3.5) we get

$$Q(x_r) \leq Q(x_{r+1}) \ \text{ and } Q(y_r) \geq Q(y_{r+1}) \tag{3.6}$$

From the condition (3.1.3) and (3.2) we get the following inequalities;

$$\begin{aligned}
&\{V(P(x_{r-1},y_{r-1}),P(x_{r-1},y_{r-1}),...,P(x_{r-1},y_{r-1}),P(x_r,y_r),kt)\}\\
\geq\ &\{V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t)*V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},P(x_{r-1},y_{r-1}),t)\\
&*V(Qx_r,Qx_r,...,Qx_r,P(x_r,y_r),t)\},
\end{aligned}$$

$$\begin{aligned}
&V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},kt)\\
\geq\ &V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t)*V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},t),
\end{aligned}$$

$$\begin{aligned}
&\{W(P(x_{r-1},y_{r-1}),P(x_{r-1},y_{r-1}),...,P(x_{r-1},y_{r-1}),P(x_r,y_r),kt)\}\\
\leq\ &\{W(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t)\diamondsuit W(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},P(x_{r-1},y_{r-1}),t)\\
&\diamondsuit W(Qx_r,Qx_r,...,Qx_r,P(x_r,y_r),t)\},
\end{aligned}$$

and

$$\begin{aligned}
&W(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},kt)\\
\leq\ &W(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t)\diamondsuit W(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},t).
\end{aligned}$$

Now, two cases arise.

Case 1: If $V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t) < V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},t)$, then

$$\begin{aligned}
V(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},kt) &\geq V(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t)\\
&\geq V(Qx_{r-2},Qx_{r-2},...,Qx_{r-2},Qx_{r-1},\frac{t}{k})\\
&\geq V(Qx_{r-2},Qx_{r-2},...,Qx_{r-2},Qx_{r-1},\frac{t}{k^2})\\
&\ .\\
&\ .\\
&\ .\\
&\geq V(Qx_0,Qx_0,...,Qx_0,Qx_1,\frac{t}{k^{r-1}})
\end{aligned}$$

and

$$\begin{aligned}
W(Qx_r,Qx_r,...,Qx_r,Qx_{r+1},kt) &\leq W(Qx_{r-1},Qx_{r-1},...,Qx_{r-1},Qx_r,t)\\
&\leq W(Qx_{r-2},Qx_{r-2},...,Qx_{r-2},Qx_{r-1},\frac{t}{k})\\
&\leq W(Qx_{r-2},Qx_{r-2},...,Qx_{r-2},Qx_{r-1},\frac{t}{k^2})\\
&\ .\\
&\ .\\
&\leq W(Qx_0,Qx_0,...,Qx_0,Qx_1,\frac{t}{k^{r-1}})
\end{aligned}$$

Then by simple induction, for all $t > 0$ and $r = 1, 2, ..., \infty$, we have that

$$V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, t) \geq V(Qx_0, Qx_0, ..., Qx_0, Qx_1, \frac{t}{k^{r-1}})$$

and

$$W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, t) \leq W(Qx_0, Qx_0, ..., Qx_0, Qx_1, \frac{t}{k^{r-1}}).$$

Thus, by condition (vi) and (xiii) of the definition of a generalized intuitionistic fuzzy metric space, for any positive integer p and real number $t > 0$, we have

$$\begin{aligned} V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+p}, t) &\geq \{V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, \frac{t}{p}) \\ &\quad *V(Qx_{r+1}, Qx_{r+1}, ..., Qx_{r+1}, Qx_{r+2}, \frac{t}{p}) \\ &\quad *... \ p \ times \ ... \\ &\quad *V(Qx_{r+p-1}, Qx_{r+p-1}, ..., Qx_{r+p-1}, Qx_{r+p}, \frac{t}{p})\} \\ &\geq V(Qx_0, Qx_0, ..., Qx_0, Qx_1, \frac{t}{pk^{r-1}}) \\ &\quad *... \ p \ times \ ... \\ &\quad *V(Qx_0, Qx_0, ..., Qx_0, Qx_1, \frac{t}{pk^{r+p-2}}) \end{aligned}$$

$$\begin{aligned} W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+p}, t) &\leq \{W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, \frac{t}{p}) \\ &\quad \diamondsuit W(Qx_{r+1}, Qx_{r+1}, ..., Qx_{r+1}, Qx_{r+2}, \frac{t}{p}) \\ &\quad \diamondsuit ... \ p \ times \ ... \\ &\quad \diamondsuit W(Qx_{r+p-1}, Qx_{r+p-1}, ..., Qx_{r+p-1}, Qx_{r+p}, \frac{t}{p})\} \\ &\leq W(Qx_0, Qx_0, ..., Qx_0, Qx_1, \frac{t}{pk^{r-1}}) \\ &\quad \diamondsuit ... \ p \ times \ ... \\ &\quad \diamondsuit W(Qx_0, Qx_0, ..., Qx_0, Qx_1, \frac{t}{pk^{r+p-2}}) \end{aligned}$$

Therefore, taking $r \to \infty$, by definition (viii) and (xv) we get

$$V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+p}, t) \geq 1 * 1 * ... * 1(ptimes)$$

and

$$W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+p}, t) \leq 0 \diamondsuit 0 \diamondsuit ... \diamondsuit 0(ptimes),$$

which implies that $\{Qx_n\}$ is a Cauchy sequence in X.

Case 2: If $V(Qx_{r-1}, Qx_{r-1}, ..., Qx_{r-1}, Qx_r, t) > V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, t)$, then

$$V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, kt) \geq V(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, t)$$

and

$$W(Qx_{r-1}, Qx_{r-1}, ..., Qx_{r-1}, Qx_r, t) < W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, t).$$

So,

$$W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, kt) \leq W(Qx_r, Qx_r, ..., Qx_r, Qx_{r+1}, t.)$$

By Lemma(2.4), we get $Qx_r = Qx_{r+1}$. Thus, there exists a positive integer m such that $r \geq m$ implies $Qx_r = Qx_m$, for all r, which shows that $\{Qx_n\}$ is a convergent sequence and so a Cauchy sequence in X.

Taking $x = y_r, y = x_r, u = y_{r-1}, v = x_{r-1}$ in (3.1.3), we get

$$\begin{aligned} &\{V(P(y_r, x_r), P(y_r, x_r), ..., \\ &P(y_r, x_r), P(y_{r-1}, x_{r-1}), kt)\} \geq \{V(Qy_r, Qy_r, ..., Qy_r, Q_u, t) \\ &\qquad *V(Qy_r, Qy_r, ..., Qy_r, P(y_r, x_r), t) \\ &\qquad *V(Qy_{r-1}, Qy_{r-1}, ..., Qy_{r-1}, P(y_{r-1}, x_{r-1}), t)\} \end{aligned}$$

and

$$\begin{aligned} &\{W(P(y_r, x_r), P(y_r, x_r), ..., \\ &P(y_r, x_r), P(y_{r-1}, x_{r-1}), kt)\} \leq \{W(Qy_r, Qy_r, ..., Qy_r, Q_u, t) \\ &\qquad \Diamond W(Qy_r, Qy_r, ..., Qy_r, P(y_r, x_r), t) \\ &\qquad \Diamond W(Qy_{r-1}, Qy_{r-1}, ..., Qy_{r-1}, P(y_{r-1}, x_{r-1}), t)\} \end{aligned}$$

So, from equation (3.1.6) we have

$$\begin{aligned} V(Qy_r, Qy_r, ..., Qy_r, Qy_{r+1}, kt) &\geq V(Qy_{r-1}, Qy_{r-1}, ..., Qy_{r-1}, Qy_r, t) \\ &\quad *V(Qy_r, Qy_r, ..., Qy_r, Qy_{r+1}, t) \end{aligned}$$

$$\begin{aligned} W(Qy_r, Qy_r, ..., Qy_r, Qy_{r+1}, kt) &\leq W(Qy_{r-1}, Qy_{r-1}, ..., Qy_{r-1}, Qy_r, t) \\ &\quad \Diamond W(Qy_r, Qy_r, ..., Qy_r, Qy_{r+1}, t) \end{aligned}$$

In the same way, $\{Qy_n\}$ is a Cauchy sequence in X. Since X is a complete space, there exist $x, y \in X$ such that

$$\lim_{r\to\infty} P(x_r, y_r) = \lim_{r\to\infty} Q(x_r) = x, \ \lim_{r\to\infty} P(y_r, x_r) = \lim_{r\to\infty} Q(y_r) = y \tag{3.7}$$

By considering condition (3.1.4) and $r \to \infty$ we have

$$V(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(Q(x_r), Q(y_r), t)) \to 1,$$

$$W(Q(P(x_r, y_r)), Q(P(x_r, y_r)), ..., Q(P(x_r, y_r)), P(Q(x_r), Q(y_r), t)) \to 0,$$

and

$$V(Q(P(y_r, x_r)), Q(P(y_r, x_r)), ..., Q(P(y_r, x_r)), P(Q(y_r), Q(x_r), t)) \to 1,$$

$$W(Q(P(y_r, x_r)), Q(P(y_r, x_r)), ..., Q(P(y_r, x_r)), P(Q(y_r), Q(x_r), t)) \to 0$$

as $r \to \infty$. By conditions (3.1.4) and (a), since P and Q are continuous, we have

$$V(Q(x), Q(x), ..., Q(x), P(x, y), t) = 1, W(Q(x), Q(x), ..., Q(x), P(x, y), t) = 0$$

and

$$V(Q(y), Q(y), ..., Q(y), P(y, x), t) = 1, W(Q(y), Q(y), ..., Q(y), P(y, x), t) = 0,$$

This implies that $P(x, y) = Q(x)$ and $P(y, x) = Q(y)$, and thus, we have proved that P and Q have a coupled coincidence point in X.

Now, suppose that condition (3.1.4) and (b) hold. Since Q is continuous and P, Q are compatible mappings, we have

$$\lim_{r\to\infty} P(Q(x_r), Q(y_r)) = \lim_{r\to\infty} Q(P(x_r, y_r)) = \lim_{r\to\infty} Q(Qx_r) = Q(x) \tag{3.8}$$

and

$$\lim_{r\to\infty} P(Q(y_r), Q(x_r)) = \lim_{r\to\infty} Q(P(y_r, x_r)) = \lim_{r\to\infty} Q(Qy_r) = Q(y) \tag{3.9}$$

By condition (vi) and (xiii) of a generalized intuitionistic fuzzy metric space, as $r \to \infty$, we get

$$\begin{aligned} V(Qx, Qx, ..., Qx, P(x,y), t) &\geq \{V(Qx, Qx, ..., Qx, Q(Qx_{r+1}), t-kt) \\ &\quad *V(Q(Qx_{r+1}), Q(Qx_{r+1}), ..., \\ &\quad Q(Qx_{r+1}), P(x,y), kt)\} \\ &= \{V(Qx, Qx, ..., Qx, Q(P(x_r, y_r)), t-kt) \\ &\quad *V(Q(P(x_r, y_r)), Q(P(x_r, y_r))... \\ &\quad V(Q(P(x_r, y_r)), P(x,y), kt))\} \\ &\geq V(Q(P(x_r, y_r)), Q(P(x_r, y_r))... \\ &\quad V(Q(P(x_r, y_r)), P(x,y), kt)) and \end{aligned}$$

$$\begin{aligned} W(Qx, Qx, , Qx, P(x,y), t) &\leq \{W(Qx, Qx, ..., Qx, Q(Qx_{r+1}), t-kt) \\ &\quad \diamondsuit W(Q(Qx_{r+1}), Q(Qx_{r+1}), ..., \\ &\quad Q(Qx_{r+1}), P(x,y), kt)\} \\ &= \{W(Qx, Qx, ..., Qx, Q(P(x_r, y_r)), t-kt) \\ &\quad \diamondsuit W(Q(P(x_r, y_r)), Q(P(x_r, y_r))... \\ &\quad W(Q(P(x_r, y_r)), P(x,y), kt))\} \\ &\leq W(Q(P(x_r, y_r)), Q(P(x_r, y_r))... \\ &\quad W(Q(P(x_r, y_r)), P(x,y), kt)) \end{aligned}$$

We get

$$\begin{aligned} V(Qx, Qx, ..., Qx, P(x,y), t) &\geq V(P(Qx_r, Qy_r), P(Qx_r, Qy_r), ..., \\ &\quad V(P(Qx_r, Qy_r), P(x,y), kt)) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} W(Qx, Qx, ..., Qx, P(x,y), t) &\leq W(P(Qx_r, Qy_r), P(Qx_r, Qy_r), ..., \\ &\quad W(P(Qx_r, Qy_r), P(x,y), kt)) \; (3.1.14) \end{aligned} \tag{3.11}$$

By using condition (3.1.3) and equations (3.8), (3.9), (3.10) and (3.11), we get

$$\begin{aligned} V(Qx, Qx, ..., Qx, P(x,y), t) &\geq \{V(Q(Qx_r), Q(Qx_r), ..., Q(Qx_r), Qx, t) \\ &\quad *V(Q(Qx_r), Q(Qx_r), ..., P(Qx_r, Qy_r), t) \\ &\quad *V(Qx, Qx, ..., Qx, P(x,y), t)\} \\ &\geq V(Qx, Qx, ..., Qx, P(x,y), t) \end{aligned}$$

and

$$\begin{aligned} W(Qx, Qx, ..., Qx, P(x, y), t) &\leq \{W(Q(Qx_r), Q(Qx_r), ..., Q(Qx_r), Qx, t) \\ &\quad \diamondsuit W(Q(Qx_r), Q(Qx_r), ..., P(Qx_r, Qy_r), t) \\ &\quad \diamondsuit W(Qx, Qx, ..., Qx, P(x, y), t)\} \\ &\leq W(Qx, Qx, ..., Qx, P(x, y), t) \end{aligned}$$

By Lemma (2.4), we have $P(x, y) = Q(x)$. Similarly, we get $P(y, x) = Q(y)$. Hence, we proved that P and Q have a coupled coincidence point in X. ∎

Example 3.1. Let $(X, \leq)$ be a partially ordered set with $X = [0, 1], a * b = \min\{a, b\}$ and $a\diamondsuit b = \max\{a, b\}$. Let $P : X \times X \to X$ and $Q : X \to X$ be two mappings defined as

$$P(x, y) = \begin{cases} \dfrac{x - y}{2} & if\ x \geq y,\ Q(x)n = x \\ 0 & if\ x < y, \end{cases}$$

This implies that P satisfies the definition of the mixed Q - monotone property. Let

$$V(x_1, x_2, x_3, ..., x_n, t) = \frac{t}{t + A(x_1, x_{2,}, x_3, ...x_n)}$$

and

$$W(x_1, x_2, x_3, ..., x_n, t) = \frac{t(x_1, x_{2,}, x_3, ...x_n)}{t + A(x_1, x_{2,}, x_3, ...x_n)},$$

where $A(x_1, x_2, ..., x_n)$ is the A- metric space defined as

$$A(x_1, x_2, ..., x_n) = |x_1 - x_2| + |x_2 - x_3| + ... + |x_{n-1} - x_n|,$$

for all $x_1, x_2, ..., x_n \in X, t > 0$.

Then $(X, V, W, *, \diamondsuit)$ be a complete generalized intuitionistic fuzzy metric space. We take $k = 1/2$ and consider the sequences $\{x_r\}, \{y_r\}$ in X defined by $x_r = \dfrac{1}{2r}, y_r = \dfrac{1}{3r}$. Since

$$\lim_{r\to\infty} P(x_r, y_r) = \lim_{r\to\infty} Q(x_r) = 0 = w \text{ (say)},$$
$$\lim_{r\to\infty} P(y_r, x_r) = \lim_{r\to\infty} Q(y_r) = 0 = w' \text{ (say)}.$$

Also, $P : X \times X \to X$ and $Q : X \to X$ are compatible mappings in X. From Theorem (3.1) we have that $Q(x) \leq Q(u)$ and $Q(y) \geq Q(v)$. This implies $x \leq u,\ y \geq v$. If we consider $x \geq y,\ u \geq v$, then we have

$$\begin{aligned} V(P(x, y), P(x, y), ..., P(x, y), P(u, v), t/2) &= \frac{t/2}{t/2 + 2|\frac{(x-y)-(u-v)}{2}|} \\ &\geq \frac{t}{t + |u + v|} \\ &= V(Q(u), Q(u), ..., Q(u), P(u, v), t) \\ &\geq \{V(Q(x), Q(x), ..., Q(x), Q(u), t) \\ &\quad *V(Q(x), Q(x), ..., Q(x), P(x, y), t) \\ &\quad *V(Q(u), Q(u), ..., Q(u), P(u, v), t)\} \end{aligned}$$

and

$$
\begin{aligned}
W(P(x,y),P(x,y),....,P(x,y),P(u,v),t/2) &= \frac{2|\frac{(x-y)-(u-v)}{2}|}{t/2+2|\frac{(x-y)-(u-v)}{2}|} \\
&\leq \frac{|u+v|}{t+|u+v|} \\
&= W(Q(u),Q(u),...,Q(u),P(u,v),t) \\
&\leq \{W(Q(x),Q(x),...,Q(x),Q(u),t) \\
&\quad \diamondsuit W(Q(x),Q(x),...,Q(x),P(x,y),t) \\
&\quad \diamondsuit W(Q(u),Q(u),...,Q(u),P(u,v),t)\}.
\end{aligned}
$$

If we consider $x < y$, $u \geq v$, then we have

$$
\begin{aligned}
V(P(x,y),P(x,y),P(u,v),t/2) &= \frac{t/2}{t/2+2|(u-v)/2|} \\
&\geq \frac{t}{t+2|u-x|} \\
&\geq V(Q(x),Q(x),...,Q(x),Q(u),t) \\
&\geq \{V(Q(x),Q(x),...,Q(x),Q(u),t) \\
&\quad *V(Q(x),Q(x),...,Q(x),P(x,y),t) \\
&\quad *V(Q(u),Q(u),...,Q(u),P(u,v),t)\}
\end{aligned}
$$

and

$$
\begin{aligned}
W(P(x,y),P(x,y),P(u,v),t/2) &= \frac{2|(u-v)/2|}{t/2+2|(u-v)/2|} \\
&\leq \frac{2|u-x|}{t+2|u-x|} \\
&\leq W(Q(x),Q(x),...,Q(x),Q(u),t) \\
&\leq \{W(Q(x),Q(x),...,Q(x),Q(u),t) \\
&\quad \diamondsuit W(Q(x),Q(x),...,Q(x),P(x,y),t) \\
&\quad \diamondsuit W(Q(u),Q(u),...,Q(u),P(u,v),t)\}
\end{aligned}
$$

If we consider $x < y$, $u < v$, then we get directly condition (3.1.3) of Theorem 3.1.

Therefore, all hypotheses of Theorem 3.1 hold. So we conclude that (w, w') is a common coupled fixed point of P and Q.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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