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ON SUBSPACES AND COMPACTNESS OF CO-SMOOTH FUZZY TOPOLOGICAL SPACES

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Abstract In almost all examples available in the literature, while introducing fuzzy concepts, certain properties like intelligence and beauty are considered on objects like students and flowers. The properties are fuzzy in nature whereas the objects are crisp in nature. One of the major reasons for introducing fuzzy mathematics is to discuss about the fuzzy properties on crisp objects. In this connection a theory called fuzzifying topology is defined and discussed by Mingsheng Ying in 1991. In this paper, the theory is extended in the context of subspaces; further a concept called α-compactness is defined and it is proved that the product of finitely many α-compact spaces is α-compact.

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1. Introduction

To study the fuzzy properties of crisp objects, the concept of fuzzifying topology is defined by Mingsheng Ying in 1991. He further defined the concepts of basis, subbasis, subspaces, product topology, compactness in [8–11]. In [16], the concept of fuzzifying topology is studied in the name of co-smooth fuzzy topology; the adjective "co" is added to represent "crisp objects". The concepts of basis, subbasis and product spaces in the context of co-smooth fuzzy topology are defined and discussed in an easier way. In fact the concept of basis defined in both the papers are same in some sense; but the second one is comparatively easy to fulfil the need of this paper. So we follow the definition in [16].

As we compare the theory developed here with the theory available in smooth fuzzy topology, we give some theory available in the literature. The concept of subspace of a

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smooth fuzzy topological space was studied by A. A. Ramadan, Werner Peeters and many others [13, 14]. In 2004, S.E Abbas gave a new definition for the subspace of a smooth fuzzy topological space. Concepts like smooth closure and smooth interior are studied by Gayyar et al.[3, 4, 12, 15].

In Section 2, we give some basic definitions and results available in the literature, which we need to develop our theory; in Section 3 we discuss the concept of subspace in the context of co-smooth fuzzy topological spaces and prove some classical results. In Section 4, we define the concept of α-openness, α-closedness, α-closure, α-interior and prove some results. In Section 5, we define α-compactness and prove that the product of finitely many α-compact spaces is α-compact.

2. Preliminary Definitions and Results

For any set X, by $\mathscr{P}(X)$ we denote the collection of all subsets of X; by $\mathbb{R}$ and $\emptyset$, we denote the set of real numbers and the empty set respectively. The infimum and supremum of $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq \mathbb{R}$ are denoted by $\underset{\lambda\in\Lambda}{\wedge} a_\lambda$ and $\underset{\lambda\in\Lambda}{\vee} a_\lambda$. As usual, we denote the interval $[0,1]$ by I and all fuzzy subsets of a set X by I^X.

Definition 2.1. [8, 16] *Let X be any set and let $\mathcal{T} : \mathscr{P}(X) \to [0,1]$ be a mapping satisfying the following conditions:*

i. $\mathcal{T}(X) = 1$
ii. $\mathcal{T}(\emptyset) = 1$
iii. $\mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ *for any two subsets A, B of X*
iv. $\mathcal{T}(\cup A_\lambda) \geq \wedge \mathcal{T}(A_\lambda)$ *for any collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of X.*

Then $\mathcal{T}$ is called a co-smooth fuzzy topology on X and the pair $(X, \mathcal{T})$ is called a co-smooth fuzzy topological space. If $A \subseteq X$, then $\mathcal{T}(A)$ is called the degree of openness of the set A in $(X, \mathcal{T})$.

Let $\mathcal{C} : \mathscr{P}(X) \to [0,1]$ be the mapping defined by $\mathcal{C}(A) = \mathcal{T}(A^c)$ where A^c is the complement of A in X; $\mathcal{C}(A)$ is called the degree of closedness of A.

Let $(X, \mathcal{T})$ be a co-smooth fuzzy topological space. If $\mathcal{C}$ is the gradation of closedness as defined in Definition 2.1, then we have $\mathcal{C}(X) = 1$, $\mathcal{C}(\emptyset) = 1$, $\mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B)$ for any two subsets of X and $\mathcal{C}(\underset{\lambda\in\Lambda}{\cap} A_\lambda) \geq \underset{\lambda\in\Lambda}{\wedge} \mathcal{C}(A_\lambda)$ for any collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of subsets of X.

If X is a crisp topological space, then by defining $\mathcal{T}(U) = 1$ if and only if U is open in X, X can be viewed as a co-smooth fuzzy topological space.

The concept of basis is defined in [8] for a given fuzzifying topology; in this case the term basis is meaningful only if a fuzzifying topology is given; of course, two necessary and sufficient conditions are proved for a given function $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ to be a basis for some fuzzifying topology. But the concept of basis defined in [16] do not need any fuzzifying topology. Further the basis defined in [16] and the basis defined in [8] generates same fuzzifying topology, but the class of bases defined in [16] is smaller and easier to handle with. So, we follow the definition of basis given in [16].

Definition 2.2. [16] *Let X be any set. Define a function $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ satisfying the following conditions:*

i. Given $x \in X$ and $\epsilon > 0$ there exists $A \subseteq X$ such that $x \in A$ and $\mathcal{B}(A) \geq 1 - \epsilon$.

ii. If $x \in A \cap B$ and $\epsilon > 0$ are given, then there exists $C \subseteq X$ such that $x \in C \subseteq A \cap B$ and $\mathcal{B}(C) \geq (\mathcal{B}(A) \wedge \mathcal{B}(B)) - \epsilon$.

then $\mathcal{B}$ is called a basis for a co-smooth fuzzy topology on X.

A collection $\{A_\lambda\}_{\lambda\in\Lambda}$ of nonempty subsets of a set A is called an inner cover for A if $A = \underset{\lambda\in\Lambda}{\cup} A_\lambda$. Let $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ be a basis for a co-smooth fuzzy topology on a set X. Define $\mathcal{T} : \mathscr{P}(X) \to [0,1]$ as $\mathcal{T}(\emptyset) = 1$ and for $\emptyset \subsetneq A \subseteq X$ define $\mathcal{T}(A) = \sup_{\Lambda\in\Gamma}\left\{\inf_{A_\lambda\in\mathfrak{C}_\Lambda}\{\mathcal{B}(A_\lambda)\}\right\}$, where $\{\mathfrak{C}_\Lambda\}_{\Lambda\in\Gamma}$ is the collection of all possible inner covers $\{A_\lambda\}_{\lambda\in\Lambda}$ of A. Then $\mathcal{T}$ is a co-smooth fuzzy topology on X which is called the co-smooth fuzzy topology generated by $\mathcal{B}$ (see [16]).

Theorem 2.3. [16] *Let $(X, \mathcal{T})$ be a co-smooth fuzzy topological space. Let $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ be a function satisfying*

i. $\mathcal{T}(A) \geq \mathcal{B}(A)$ for all $A \subseteq X$.
ii. If $A \subseteq X$, $x \in A$ and $\epsilon > 0$, then there exists $B \subseteq X$ such that $x \in B \subseteq A$ and $\mathcal{B}(B) \geq \mathcal{T}(A) - \epsilon$.

Then $\mathcal{B}$ is a basis for the co-smooth fuzzy topology $\mathcal{T}$ on X.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two co-smooth fuzzy topological spaces. Then the function $\mathcal{B} : \mathscr{P}(X \times Y) \to [0,1]$ defined as

$$\mathcal{B}(E) = \begin{cases} \inf\{\mathcal{T}(E_1), \mathcal{T}'(E_2)\} & \text{if } \ E = E_1 \times E_2 \\ 0 & \text{otherwise} \end{cases}$$

is a basis for a co-smooth fuzzy topology on $X \times Y$ and the co-smooth fuzzy topology generated by $\mathcal{B}$ is called the co-smooth fuzzy product topology of $\mathcal{T}$ and $\mathcal{T}'$ on $X \times Y$(see [16]).

Theorem 2.4. [16] *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two co-smooth fuzzy topological spaces. Let $\mathcal{B}_X, \mathcal{B}_Y$ be bases for the topologies $\mathcal{T}_X, \mathcal{T}_Y$. Then the function $\mathcal{B}_{X\times Y} : \mathscr{P}(X \times Y) \to [0,1]$ defined as*

$$\mathcal{B}_{X\times Y}(E) = \begin{cases} \inf\{\mathcal{B}_X(A), \mathcal{B}_Y(B)\} & \textit{if } \ E = A \times B \\ 0 & \textit{otherwise} \end{cases}$$

is a basis for the co-smooth product topology on $X \times Y$.

Definition 2.5. [16] *A subset A of a co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to be α-open, $\alpha \in [0,1)$ if $\mathcal{T}_X(A) > \alpha$ and is said to be α-closed if the set $X - A$ is α-open. That is, if A is α-closed, then $\mathcal{C}_X(A) > \alpha$.*

Definition 2.6. [16] *A co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to be α-Hausdorff if for each pair x, y of distinct points of X, there exist disjoint α-open sets U and V containing x and y respectively. Hausdorffness of a topological space is defined as the supremum of all α such that $(X, \mathcal{T})$ is α-Hausdorff.*

If $(X, \mathcal{T})$ is a (crisp) topological space, then it can be viewed as a co-smooth fuzzy topological space as mentioned below Definition 2.1. If $(X, \mathcal{T})$ is Hausdorff, in the crisp sense, then it is α-Hausdorff for all α and hence its Hausdorffness is 1.

Definition 2.7. [13] *Let* $(X, \mathcal{T})$ *be a smooth topological space and let* $Y \subseteq X$*. Define the gradation mapping* $\mathcal{T}_Y : I^Y \to I$ *as*

$$\forall \mu \in I^Y : \mathcal{T}_Y(\mu) = \vee\{\mathcal{T}(\nu) : \nu \in I^X, \nu/Y = \mu\}.$$

Then $\mathcal{T}_Y$ *is a smooth fuzzy topology on* Y*. We call* $(Y, \mathcal{T}_Y)$ *a smooth fuzzy topological subspace.*

3. Co-smooth Fuzzy Subspaces

We start with the definition of the subspace co-smooth fuzzy topology.

Definition 3.1. [9] *Let* $(X, \mathcal{T})$ *be a co-smooth fuzzy topological space and let* $Y \subseteq X$*. Then the function* $\mathcal{T}_Y : \mathscr{P}(Y) \to [0,1]$ *defined as*

$$\mathcal{T}_Y(A) = \sup\{\mathcal{T}_X(B)/B \cap Y = A, B \subseteq X\}$$

is a co-smooth fuzzy topology on Y *called the subspace co-smooth fuzzy topology induced over* Y *by* $\mathcal{T}$*, with this topology* Y *is called a co-smooth fuzzy subspace of* X*.*

Let $(X, \mathcal{T}_X)$ be a co-smooth fuzzy topological space and let $\mathcal{C}_X$ be the gradation of closedness with respect to $\mathcal{T}_X$. Let $Y \subseteq X$ be with subspace co-smooth fuzzy topology $\mathcal{T}_Y$ and let $\mathcal{C}_Y$ be the gradation of closedness on Y, with respect to $\mathcal{T}_Y$. Throughout this section the labels $(X, \mathcal{T}_X)$, $\mathcal{C}_X$, $\mathcal{T}_X$, Y, $\mathcal{T}_Y$ and $\mathcal{C}_Y$ are used in this meaning, unless otherwise stated.

Theorem 3.2. *Let* $\mathcal{B}_X$ *be a basis for a co-smooth fuzzy topological space* $(X, \mathcal{T}_X)$*. For* $A \in \mathscr{P}(Y)$ *define*

$$\mathcal{B}_Y(A) = \sup\{\mathcal{B}_X(B)/B \cap Y = A, B \subseteq X\}.$$

Then $\mathcal{B}_Y$ *is a basis for the subspace co-smooth fuzzy topology on* Y*.*

Proof. For any $A \subseteq Y$, we have

$$\begin{aligned} \mathcal{T}_Y(A) &= \sup\{\mathcal{T}_X(B)/B \subseteq X, B \cap Y = A\} \\ &\geq \sup\{\mathcal{B}_X(B)/B \subseteq X, B \cap Y = A\} \\ &= \mathcal{B}_Y(A). \end{aligned}$$

Let $y \in A \subseteq Y$ and $\epsilon > 0$. Then by definition of $\mathcal{T}_Y$, there exists $C \subseteq X$ such that $C \cap Y = A$ and $\mathcal{T}_X(C) \geq \mathcal{T}_Y(A) - \frac{\epsilon}{2}$. Since $\mathcal{B}_X$ is a basis for the co-smooth fuzzy topology $\mathcal{T}_X$, there exist an inner cover $\mathfrak{C}_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$ of C such that

$$\inf_{E_\lambda \in \mathfrak{C}_\Lambda} \{\mathcal{B}_X(E_\lambda)\} \geq \mathcal{T}_X(C) - \frac{\epsilon}{2}.$$

Now since, $C \cap Y = A$, $\mathfrak{C}_\Lambda = \{E_\lambda\}_{\lambda \in \Lambda}$ is an inner cover for C; we have, $(\cup E_\lambda) \cap Y = A$. Thus there exists some $E_0 \in \{E_\lambda\}_{\lambda \in \Lambda}$, such that $y \in E_0$ and $E_0 \cap Y \subseteq A$. Therefore by letting $B = E_0 \cap Y$, it clearly follows that $y \in B$ and $\mathcal{B}_Y(B) \geq \mathcal{B}_X(E_0)$. But,

$$\mathcal{T}_Y(A) - \frac{\epsilon}{2} \leq \mathcal{T}_X(C) \leq \inf_{E_\lambda \in \mathfrak{C}_\Lambda} \{\mathcal{B}_X(E_\lambda)\} + \frac{\epsilon}{2}.$$

This implies that

$$\begin{aligned}\mathcal{T}_Y(A) &\leq \inf_{E_\lambda \in \mathfrak{C}_\Lambda}\{\mathcal{B}_X(E_\lambda)\} + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \inf_{E_\lambda \in \mathfrak{C}_\Lambda}\{\mathcal{B}_X(E_\lambda)\} + \epsilon \\ &\leq \mathcal{B}_X(E_0) + \epsilon \\ &\leq \mathcal{B}_Y(B) + \epsilon\end{aligned}$$

Thus there exists $B \subseteq Y$ such that $y \in B \subseteq A$ and $\mathcal{B}_Y(B) \geq \mathcal{T}_Y(A) - \epsilon$ and hence by Theorem 2.3, $\mathcal{B}_Y$ is a basis for the subspace co-smooth fuzzy topology on Y. ∎

Theorem 3.3. *Let $A \subseteq Y \subseteq X$. If $\mathcal{T}_Y(A) \geq \alpha$ and $\mathcal{T}_X(Y) \geq \alpha$ then $\mathcal{T}_X(A) \geq \alpha$.*

Proof. Let $A \subseteq Y$, $\mathcal{T}_Y(A) \geq \alpha$ and $\epsilon > 0$; then by definition of $\mathcal{T}_Y$, there exists $B \subseteq X$ such that $B \cap Y = A$ and $\mathcal{T}_X(B) \geq \mathcal{T}_Y(A) - \epsilon \geq \alpha - \epsilon$. Now, $\mathcal{T}_X(A) = \mathcal{T}_X(B \cap Y) \geq \mathcal{T}_X(B) \wedge \mathcal{T}_X(Y) \geq (\alpha - \epsilon) \wedge \alpha = \alpha - \epsilon$. Since this is true for every $\epsilon > 0$, we get that $\mathcal{T}_X(A) \geq \alpha$. ∎

As the converse above theorem is obvious, we have "If $A \subseteq Y$ and $\mathcal{T}_X(Y) \geq \alpha$, then $\mathcal{T}_X(A) \geq \alpha$ if and only if $\mathcal{T}_Y(A) \geq \alpha$."

Corollary 3.4. *If $A \subseteq Y \subseteq X$, then $\mathcal{T}_X(A) \geq \mathcal{T}_Y(A) \wedge \mathcal{T}_X(Y)$.*

Proof. Let $\mathcal{T}_Y(A) \wedge \mathcal{T}_X(Y) = \alpha$, then $\mathcal{T}_Y(A) \geq \alpha$ and $\mathcal{T}_X(Y) \geq \alpha$. Hence by Theorem 3.3, we have $\mathcal{T}_X(A) \geq \alpha$. ∎

Both equality and inequality may hold in the above corollary.

Example 3.5. *Let $X = \{1, 2, 3, 4\}$, $Y = \{1, 2\}$. For $E \in \mathscr{P}(X)$ define*

$$\mathcal{T}_{1_X}(E) = \begin{cases} 1 & \text{if } E = X \text{ or } E = \emptyset \\ \frac{1}{2} & \text{if } E \in \{\{1,2\}, \{1,3\}, \{1,2,3\}, \{1\}\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{T}_{2_X}(E) = \begin{cases} \mathcal{T}_{1_X}(E) & \text{if } E \neq \{1\} \\ \frac{3}{4} & \text{if } E = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

Then, clearly $\mathcal{T}_{1_X}$ and $\mathcal{T}_{2_X}$ are co-smooth fuzzy topologies on X. Now by the definition of subspace co-smooth fuzzy topology, we have

$$\mathcal{T}_{1_Y}(E) = \begin{cases} 1 & \text{if } E = Y \text{ or } E = \emptyset \\ \frac{1}{2} & \text{if } E = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{T}_{2_Y}(E) = \begin{cases} 1 & \text{if } E = Y \text{ or } E = \emptyset \\ \frac{3}{4} & \text{if } E = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{1\}$, then $\mathcal{T}_{1_X}(A) = \frac{1}{2}$, $\mathcal{T}_{1_X}(Y) = \frac{1}{2}$, $\mathcal{T}_{1_Y}(A) = \frac{1}{2}$ $\mathcal{T}_{2_X}(A) = \frac{3}{4}$, $\mathcal{T}_{2_X}(Y) = \frac{1}{2}$ and $\mathcal{T}_{2_Y}(A) = \frac{3}{4}$. Thus, we have $\mathcal{T}_{1_X}(A) = \mathcal{T}_{1_Y}(A) \wedge \mathcal{T}_{1_X}(Y)$ and $\mathcal{T}_{2_X}(A) > \mathcal{T}_{2_Y}(A) \wedge \mathcal{T}_{2_X}(Y)$..

Theorem 3.6. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two co-smooth fuzzy topological spaces. Let $(A, \mathcal{T}_A)$, $(B, \mathcal{T}_B)$ be co-smooth fuzzy subspaces of X and Y. Then the co-smooth fuzzy product topology on $A \times B$ is same as the co-smooth fuzzy topology on $A \times B$ inherits as the co-smooth fuzzy subspace of $X \times Y$.*

Proof. Let $\mathcal{B}_X$, $\mathcal{B}_Y$ be bases for the co-smooth fuzzy topologies $\mathcal{T}_X$, $\mathcal{T}_Y$. For $G \subseteq A$ and $H \subseteq B$, let $\mathcal{B}_A(G) = \sup\{\mathcal{B}_X(C)/C \cap A = G, C \subseteq X\}$ and $\mathcal{B}_B(H) = \sup\{\mathcal{B}_Y(D)/D \cap B = H, D \subseteq Y\}$. Then by Theorem 3.2, $\mathcal{B}_A$, $\mathcal{B}_B$ are bases for the co-smooth fuzzy topologies $\mathcal{T}_A$, $\mathcal{T}_B$. Let $\mathcal{B}^p_{A\times B}$ be the function from $\mathscr{P}(A \times B)$ to $[0, 1]$ defined as follows:

$$\mathcal{B}^p_{A\times B}(E) = \begin{cases} \inf\{\mathcal{B}_A(G), \mathcal{B}_B(H)\} & \text{if} \quad E = G \times H,\ G \subseteq A,\ H \subseteq B \\ 0 & \text{otherwise} \end{cases}.$$

Then by Theorem 2.4, $\mathcal{B}^p_{A\times B}$ is a basis for the co-smooth fuzzy product topology on $A \times B$. Let $\mathcal{B}^p_{X\times Y}$ be the function from $\mathscr{P}(X \times Y)$ to $[0, 1]$ defined as follows:

$$\mathcal{B}^p_{X\times Y}(E) = \begin{cases} \inf\{\mathcal{B}_X(C), \mathcal{B}_Y(D)\} & \text{if} \quad E = C \times D,\ C \subseteq X,\ D \subseteq Y \\ 0 & \text{otherwise} \end{cases}.$$

Then by Theorem 2.4, $\mathcal{B}^p_{X\times Y}$ is a basis for the co-smooth fuzzy product topology on $X \times Y$. Let $\mathcal{B}^s_{A\times B}$ be the function from $\mathscr{P}(A \times B)$ to $[0, 1]$ defined as follows:

$$\mathcal{B}^s_{A\times B}(E) = \sup\{\mathcal{B}^p_{X\times Y}(F)/F \cap (A \times B) = E,\ F \subseteq X \times Y\}.$$

Then by Theorem 3.2, $\mathcal{B}^s_{A\times B}$ is a basis for the subspace co-smooth fuzzy topology on $A \times B$. Now we claim that $\mathcal{B}^s_{A\times B}(E) = \mathcal{B}^p_{A\times B}(E)$ for all subsets E of $A \times B$. Let $E \subseteq A \times B$. Suppose $E = G \times H$ for some $G \subseteq A$, $H \subseteq B$. Then,

$$\begin{aligned} \mathcal{B}^p_{A\times B}(E) &= \inf\{\mathcal{B}_A(G), \mathcal{B}_B(H)\} \\ &= \inf\{\sup\{\mathcal{B}_X(C)/C \subseteq X, C \cap A = G\}, \\ &\qquad \sup\{\mathcal{B}_Y(D)/D \subseteq Y, D \cap B = H\}\} \\ &= \sup\{\inf_{(C,D)}\{\mathcal{B}_X(C), \mathcal{B}_Y(D)\}\} \\ &= \sup\{\mathcal{B}^p_{X\times Y}(C \times D)\} \end{aligned}$$

Thus we get

$$\mathcal{B}^p_{A\times B}(E) = \sup\{\mathcal{B}^p_{X\times Y}(C \times D)\}$$

where the supremum is taken over all possible pairs (C, D), such that $C \subseteq X$, $D \subseteq Y$ and $C \cap A = G$, $D \cap B = H$.

Now we note that if $E = F \cap (A \times B)$ for some $F \subseteq X \times Y$ and $F \neq C \times D$ for any pair (C, D) where $C \subseteq X$ and $D \subseteq Y$ then, by the definition of $\mathcal{B}^p_{X\times Y}$, we get that $\mathcal{B}^p_{X\times Y}(F) = 0$. Thus to find $\mathcal{B}^s_{A\times B}(E)$ it is enough to find the supremum of $\{\mathcal{B}^p_{X\times Y}(F)\}$ where F is of the form $C \times D$ for some $C \subseteq X$, $D \subseteq Y$. That is,

$$\mathcal{B}^s_{A\times B}(E) = \sup\{\mathcal{B}^p_{X\times Y}(C \times D)\},$$

where the supremum is taken over all possible pairs (C, D), such that $C \subseteq X$, $D \subseteq Y$ and $(C \times D) \cap (A \times B) = G \times H = E$. Hence, when $E = G \times H$ for some $G \subseteq A$, $H \subseteq B$, we have

$$\mathcal{B}^p_{A\times B}(E) = \mathcal{B}^s_{A\times B}(E).$$

Suppose that E is not of the form $G \times H$ for any pair (G, H), $G \subseteq A$, $H \subseteq B$. Then by the definition of $\mathcal{B}^p_{A\times B}$, we have, $\mathcal{B}^p_{A\times B}(E) = 0$. Suppose there exist a subset F of $X \times Y$ of the form $C \times D$, then

$$E = F \cap (A \times B) = (C \times D) \cap (A \times B) = (C \cap A) \times (D \cap B),$$

which is a contradiction to our assumption that E is not of the form $G \times H$ for any pair (G, H). Thus there is no subset F of $X \times Y$ is of the form $C \times D$ for some $C \subseteq X$ and $D \subseteq Y$ such that $F \cap (A \times B) = E$. Thus it follows that $\mathcal{B}^s_{A\times B}(E) = 0$. ∎

Ramadan [14] proved the following theorem already in the context of smooth fuzzy topological spaces. However proof of the theorem is different from the one given by Ramadan.

Theorem 3.7. *Let $(X, \mathcal{T}_X)$ be a co-smooth fuzzy topological space and let $B \subset A \subset X$. Let $\mathcal{T}_{A_X}$ and $\mathcal{T}_{B_X}$ be subspace co-smooth fuzzy topologies on A and B induced by $\mathcal{T}_X$; let $\mathcal{T}_{B_A}$ be the subspace co-smooth fuzzy topology on B induced by $\mathcal{T}_{A_X}$ respectively. Then the two co-smooth fuzzy topologies $\mathcal{T}_{B_X}$ and $\mathcal{T}_{B_A}$ on B are the same.*

Proof. Let $\mathcal{B}_X$ be a basis for $\mathcal{T}_X$. Let $\mathcal{B}_{A_X}$, $\mathcal{B}_{B_X}$ and $\mathcal{B}_{B_A}$ be the bases for the co-smooth fuzzy subspace topologies $\mathcal{T}_{A_X}$, $\mathcal{T}_{B_X}$ and $\mathcal{T}_{B_A}$ as defined in Theorem 3.2. Now we prove that $\mathcal{B}_{B_X} = \mathcal{B}_{B_A}$. Let $E \subseteq B$, then by the definitions of $\mathcal{B}_{B_X}$ and $\mathcal{B}_{B_A}$ we have

$$\mathcal{B}_{B_X}(E) = \sup\{\mathcal{B}_X(F)/F \cap B = E, F \subseteq X\}$$

and

$$\mathcal{B}_{B_A}(E) = \sup\{\mathcal{B}_{A_X}(G)/G \cap B = E, G \subseteq A\}.$$

Let $\mathcal{B}_{B_X}(E) = \lambda$ and $\epsilon > 0$, then there exists a $F \subseteq X$ such that $\mathcal{B}_X(F) \geq \lambda - \epsilon$ and $F \cap B = E$. As $F \cap B = E$, it clearly follows that $(F \cap A) \cap B = E$ and $F \cap A \subseteq A$. Since $\mathcal{B}_X(F) \geq \lambda - \epsilon$ and by the definition of $\mathcal{B}_{A_X}$, it follows that $\mathcal{B}_{A_X}(F \cap A) \geq \lambda - \epsilon$. Let $G = F \cap A$, then we have $\mathcal{B}_{A_X}(G) \geq \lambda - \epsilon$, $G \cap B = E$ and $G \subseteq A$. Thus for each $\epsilon > 0$ we can find $G \subseteq A$ such that $\mathcal{B}_{A_X}(G) \geq \lambda - \epsilon$ and $G \cap B = E$. This implies that

$$\mathcal{B}_{B_A}(E) = \sup\{\mathcal{B}_{A_X}(G)/G \cap B = E, G \subseteq A\} \geq \lambda.$$

To prove the equality, suppose $\mathcal{B}_{B_A}(E) > \lambda$. Then

$$\sup\{\mathcal{B}_{A_X}(G)/G \cap B = E, G \subseteq A\} > \lambda.$$

Let $\mathcal{B}_{B_A}(E) = \delta$. Choose $\epsilon > 0$ such that $\delta > \delta - \epsilon > \lambda$, then by definition of $\mathcal{B}_{B_A}$, we can find $G \subseteq A$ such that $\mathcal{B}_{A_X}(G) \geq \delta - \frac{\epsilon}{2}$ and $G \cap B = E$. Now by the definition of $\mathcal{B}_{A_X}$, we can find $H \subseteq X$ such that $H \cap A = G$ and $\mathcal{B}_X(H) = \delta - \frac{\epsilon}{2} - \frac{\epsilon}{2}$. Now,

$$E = G \cap B = (H \cap A) \cap B = (H \cap B) \cap A = H \cap B.$$

Thus there exists a $H \subseteq X$ such that $H \cap B = E$ and

$$\mathcal{B}_X(H) = \delta - \epsilon > \lambda.$$

This leads to a contradiction to the fact that $\mathcal{B}_{B_X}(E) = \lambda$ and hence it follows that $\mathcal{B}_{B_A}(E) = \lambda$. Therefore $\mathcal{B}_{B_X} = \mathcal{B}_{B_A}$. ∎

Theorem 3.8. *For all $A \subseteq Y \subseteq X$,*

$$\mathcal{C}_Y(A) = \sup\{\mathcal{C}_X(F)/F \cap Y = A, F \subseteq X\}.$$

Proof. Let $A \subseteq Y$. As $\mathcal{T}_Y(Y-A) = \sup\{\mathcal{T}_X(E)/E \cap Y = Y - A, E \subseteq X\}$ and $\mathcal{C}_Y(A) = \mathcal{T}_Y(Y-A)$, it is enough to prove that

$$\{\mathcal{C}_X(F)/F \cap Y = A, F \subseteq X\} = \{\mathcal{T}_X(E)/E \cap Y = Y - A, E \subseteq X\}.$$

Let $\delta \in \{\mathcal{T}_X(E)/E \cap Y = Y - A, E \subseteq X\}$, then there exists E such that $\mathcal{T}_X(E) = \delta$ and $E \cap Y = Y - A$. Let $F = X - E$, then $F \cap Y = A$ and $\mathcal{C}_X(F) = \mathcal{T}_X(X-F) = \mathcal{T}_X(E) = \delta$. Thus,

$$\{\mathcal{T}_X(E)/E \cap Y = Y - A, E \subseteq X\} \subseteq \{\mathcal{C}_X(F)/F \cap Y = A, F \subseteq X\}.$$

On the other hand, let $\delta \in \{\mathcal{C}_X(F)/F \cap Y = A, F \subseteq X\}$, then there exists F such that $\mathcal{C}_X(F) = \delta$ and $F \cap Y = A$. Let $E = X - F$, then $E \cap Y = Y - A$ and $\mathcal{T}_X(E) = \mathcal{C}_X(X-E) = \mathcal{C}_X(F) = \delta$. Thus follows that

$$\{\mathcal{C}_X(F)/F \cap Y = A, F \subseteq X\} \subseteq \{\mathcal{T}_X(E)/E \cap Y = Y - A, E \subseteq X\}.$$

∎

Note that Werner Peeters [13] proved the above theorem in the context of smooth fuzzy topological spaces. But the theorem does not hold in smooth fuzzy subspace topology according to the definition given by S. E Abbas in [1]. We now give the definition of a subspace smooth fuzzy topology verbatically as in [1] and show that the theorem is not true with that definition by an example.
For $\mu \in I^X$, let $\mathscr{A}_\mu = \{U \in I^X : U \leq \mu\}$.

Definition 3.9. *[1] Let $(X, \mathcal{T})$ be a smooth fuzzy topological space and $\mu \in I^X$. The mapping $\mathcal{T}_\mu : \mathscr{A}_\mu \to [0,1]$ defined by*

$$\mathcal{T}_\mu(U) = \vee\{\mathcal{T}(V) : V \in I^X, V \wedge \mu = U\}$$

is a smooth μ-topology induced over μ by $\mathcal{T}$. For any $U \in \mathscr{A}_\mu$, the number $\mathcal{T}_\mu(U)$ is called the μ-openness degree of U.

Example 3.10. *Let $X = \{1,2,3,4\}$ define $\mathcal{T} : I^X \to [0,1]$ as follows*

$$\mathcal{T}(A) = \begin{cases} 1 & \text{if } A = 1_X \text{ or } A = 0_X \\ \frac{1}{2} & \text{if } A(x) = \frac{1}{2} \text{ for all } x \\ \frac{3}{4} & \text{if } A(x) = \frac{3}{4} \text{ for all } x \\ \frac{1}{4} & \text{if } A(x) = \frac{1}{4} \text{ for all } x \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $\mathcal{T}$ is a smooth fuzzy topology on X. Let $\mathcal{C}$ be the gradation of closedness with respect to $\mathcal{T}$ defined as $\mathcal{C}(U) = \mathcal{T}(1_X - U)$. Then

$$\mathcal{C}(A) = \begin{cases} 1 & \text{if } A = 1_X \text{ or } A = 0_X \\ \frac{1}{2} & \text{if } A(x) = \frac{1}{2} \text{ for all } x \\ \frac{1}{4} & \text{if } A(x) = \frac{3}{4} \text{ for all } x \\ \frac{3}{4} & \text{if } A(x) = \frac{1}{4} \text{ for all } x \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mu(x) = \frac{3}{4}$ for all $x \in X$. Then by the above definition we get $\mathcal{T}_\mu : \mathscr{A}_\mu \to [0,1]$ as

$$\mathcal{T}_\mu(U) = \begin{cases} 1 & \text{if } A = \mu \text{ or } A = 0_X \\ \frac{1}{2} & \text{if } A(x) = \frac{1}{2} \text{ for all } x \\ \frac{1}{4} & \text{if } A(x) = \frac{1}{4} \text{ for all } x \\ 0 & \text{otherwise.} \end{cases}$$

Now, for $U(x) = \frac{1}{2}$ for all x, by Theorem 3.8, we get

$$\mathcal{C}_\mu(U) = \vee\{\mathcal{C}(F)/F \wedge \mu = U, F \in I^X\} = \frac{1}{2}.$$

But by the definition of gradation of closedness with respect to $\mathcal{T}_\mu$,

$$\mathcal{C}_\mu(U) = \mathcal{T}_\mu(\mu - U) = \frac{1}{4} \neq \frac{1}{2}.$$

Lemma 3.11. *Let A be a subset of X, then $\mathcal{T}_Y(A \cap Y) \geq \mathcal{T}_X(A)$ and $\mathcal{C}_Y(A \cap Y) \geq \mathcal{C}_X(A)$.*

Theorem 3.12. *For any $A \subseteq Y$, if $\mathcal{C}_Y(A) \geq \alpha$ and $\mathcal{C}_X(Y) \geq \alpha$, then $\mathcal{C}_X(A) \geq \alpha$ and hence if $A \subseteq Y$, then $\mathcal{C}_X(A) \geq \mathcal{C}_Y(A) \wedge \mathcal{C}_X(Y)$.*

The proof is similar to the proof of Theorem 3.3.

4. α-Closed Sets and α-Limit Points

In this section we define the concepts of α-closure and α-interior of a set and prove some results.

Theorem 4.1. *Let $A \subseteq Y \subseteq X$. Then A is α-closed in Y if and only if it is equal to the intersection of an α-closed subset of X with Y.*

Proof. Let A be an α-closed set in Y. Then

$$\mathcal{C}_Y(A) = \sup\{\mathcal{C}_X(F)/F \cap Y = A, F \subseteq X\} = \beta$$

for some $\beta > \alpha$. Let us choose $\epsilon > 0$ such that $\beta - \epsilon > \alpha$. Then there exists $F \subseteq X$ such that $\mathcal{C}_X(F) > \beta - \epsilon > \alpha$ and $F \cap Y = A$. Thus A is equal to the intersection of an α-closed subset of X with Y.

Conversely, let $B \subseteq X$ be such that $\mathcal{C}_X(B) > \alpha$ and $B \cap Y = A$. Since $\mathcal{C}_X(B) > \alpha$, we have $\mathcal{T}_X(X - B) > \alpha$. Then by Lemma 3.11 we get that $\mathcal{T}_Y((X - B) \cap Y) > \alpha$. But $(X - B) \cap Y = Y - A$. This implies that $\mathcal{T}_Y(Y - A) > \alpha$ and hence $\mathcal{C}_Y(A) > \alpha$. ∎

Definition 4.2. *Let $(X, \mathcal{T}_X)$ be a co-smooth fuzzy topological space. Let $\mathcal{C}_X$ be the gradation of closedness with respect to the topology $\mathcal{T}_X$. Let A be a subset of X and $\alpha \in [0,1)$. The α-closure of A is defined as the intersection of all α-closed sets containing A and the α-interior of A is defined as the union of all α-open sets contained in A.*

The α-closure and α-interior of A with respect to the space X is denoted by $Cl_{\alpha,X}(A)$ and $Int_{\alpha,X}(A)$ respectively. We note that the α-closure of a set need not be α-closed; however its closedness is greater than or equal to α. Indeed,

$$Cl_{\alpha,X}(A) = \cap\{B \ / \ A \subseteq B, \mathcal{C}(B) > \alpha, B \subseteq X\};$$

thus $\mathcal{C}_X(Cl_{\alpha,X}(A)) = \mathcal{C}_X(\cap B) \geq \wedge \mathcal{C}_X(B) \geq \alpha$. Also we note that the α-interior of a set need not be α-open and its openness is greater than or equal to α.

Theorem 4.3. *Let $A \subseteq X$ and A be an α-closed set, then the α-closure is A.*

The proof of the theorem follows from the definition of α-closure of a set. Note that the converse of the above theorem is not true. For example, let $X = \mathbb{R}$ and define $\mathcal{T} : \mathscr{P}(X) \to [0,1]$ as follows:

$$\mathcal{T}(E) = \begin{cases} 1 & \text{if } E = X \text{ or } E = \emptyset \\ \frac{1}{2} + \frac{1}{n} & \text{if } E = (-\infty, -\frac{1}{n}) \cup (\frac{1}{n}, \infty),\ n \in \mathbb{N} \\ \frac{1}{2} & \text{if } E = (-\infty, 0) \cup (0, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a co-smooth fuzzy topology on X and the gradation of closedness with respect to $\mathcal{T}$ is given by

$$\mathcal{C}(E) = \begin{cases} 1 & \text{if } E \in \{X, \emptyset\} \\ \frac{1}{2} + \frac{1}{n} & \text{if } E = [-\frac{1}{n}, \frac{1}{n}],\ n \in \mathbb{N} \\ \frac{1}{2} & \text{if } E = \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Now take $A = \{0\}$, then clearly $\frac{1}{2}$-closure of A is A whereas A is not $\frac{1}{2}$-closed.

Similarly the α-interior of an α-open set A is A itself and not conversely, for example the $\frac{1}{2}$-interior of the set $E = (-\infty, 0) \cup (0, \infty)$ is E whereas E is not $\frac{1}{2}$-open.

Theorem 4.4. *Let $A \subseteq Y \subseteq X$. Then $Cl_{\alpha,Y}(A) = Cl_{\alpha,X}(A) \cap Y$.*

Proof. Let $\mathscr{C}_{A_X} = \{B \ / \ A \subseteq B, \mathcal{C}_X(B) > \alpha, B \subseteq X\}$. Then for each $B \in \mathscr{C}_{A_X}$ the set $B \cap Y \in \{D \ / \ A \subseteq D, \mathcal{C}_Y(D) > \alpha, D \subseteq Y\}$. This implies that

$$Cl_{\alpha,Y}(A) \subseteq \bigcap_{B \in \mathscr{C}_{A_X}} (B \cap Y) = (\bigcap_{B \in \mathscr{C}_{A_X}} B) \cap Y = Cl_{\alpha,X}(A) \cap Y.$$

On the other hand, let $x \notin Cl_{\alpha,Y}(A)$. If $x \notin Y$, then clearly $x \notin Cl_{\alpha,X}(A) \cap Y$. Suppose $x \in Y$, then we claim that $x \notin Cl_{\alpha,X}(A)$. For, since $x \notin Cl_{\alpha,Y}(A)$, we have

$$x \notin \cap\{B/A \subseteq B, \mathcal{C}_Y(B) > \alpha, B \subseteq Y\}.$$

This implies that there exists $B \in \{B/A \subseteq B, \mathcal{C}_Y(B) > \alpha, B \subseteq Y\}$ such that $x \notin B$. But since $\mathcal{C}_Y(B) > \alpha$, we have

$$\sup\{\mathcal{C}_X(D)/D \cap Y = B, D \subseteq X\} > \alpha.$$

This implies that there exists $D \subseteq X$, $D \cap Y = B$ such that $\mathcal{C}_X(D) > \alpha$. Now since $x \notin B$ and $x \in Y$, we have $x \notin D$. Thus there exist an α-closed subset D of X, such that $D \cap Y = B \supseteq A$. Therefore

$$x \notin \cap\{D/A \subseteq D, \mathcal{C}_X(D) > \alpha, D \subseteq X\}.$$

This implies that $x \notin Cl_{\alpha,X}(A)$ and hence $Cl_{\alpha,X}(A) \cap Y \subseteq Cl_{\alpha,Y}(A)$. ∎

Theorem 4.5. *Let $A \subseteq X$ and let $\mathcal{B}$ be a basis for $\mathcal{T}_X$. Then*

(a) $x \in Cl_{\alpha,X}(A)$ if and only if every α-open set containing x intersects A.

(b) $x \in Cl_{\alpha,X}(A)$ if and only if every $B \subseteq X$ containing x such that $\mathcal{B}(B) > \alpha$ intersects A.

Proof. To prove (a), let us assume that $x \notin Cl_{\alpha,X}(A)$. Then

$$x \notin \cap\{D/A \subseteq D, \mathcal{C}_X(D) > \alpha, D \subseteq X\}.$$

Thus there exists D such that $A \subseteq D$, $\mathcal{C}_X(D) > \alpha$ and $x \notin D$. Let $U = X - D$. Then since $\mathcal{C}_X(D) > \alpha$ and $x \notin D$, we have $\mathcal{T}_X(U) = \mathcal{T}_X(X - D) > \alpha$ and $x \in U$. Thus there exists a α-open set U containing x such that $U \cap A = \emptyset$.

Conversely, assume that there exists a α-open set U containing x such that $U \cap A = \emptyset$. Let $B = X - U$, then $\mathcal{C}_X(B) > \alpha$ and $A \subseteq B$. But by the definition of $Cl_{\alpha,X}(A)$, we have $Cl_{\alpha,X}(A) \subseteq B$. Now since $x \in U$ and $B = X - U$, it follows that $x \notin Cl_{\alpha,X}(A)$.

To prove (b), let $x \in Cl_{\alpha,X}(A)$. Let $B \subseteq X$, $x \in B$ and $\mathcal{B}(B) > \alpha$. Then $\mathcal{T}_X(B) > \alpha$ and hence B is an α-open set containing x. Thus by (a), B intersects A.

Conversely, let $x_0 \in X$ and assume that every subset B containing x_0 with $\mathcal{B}(B) > \alpha$ intersect A. By (a), it is enough to prove that every α-open set containing x intersects A. Let U be an α-open set, then $\mathcal{T}_X(U) > \alpha$. Let $\epsilon > 0$ be such that $\mathcal{T}_X(U) - \epsilon > \alpha$, then there exists an inner cover $\{B_\lambda\}_{\lambda\in\Lambda}$ such that

$$\inf\{\mathcal{B}(B_\lambda)\} \geq \mathcal{T}_X(U) - \epsilon > \alpha.$$

Thus there exists a $B_{\lambda_0} \in \{B_\lambda\}_{\lambda\in\Lambda}$ containing x_0 and $\mathcal{B}(B_{\lambda_0}) > \alpha$. But by our assumption B_{λ_0} intersects A and hence U intersects A. Thus, by (a), it follows that $x_0 \in Cl_{\alpha,X}(A)$. ∎

Definition 4.6. *A point $x \in X$ is called an α-limit point of a set A if every α-open set containing x contains a point of A other than x.*

Theorem 4.7. *Let $A \subseteq X$ and let A'_α be the set of all α-limit points of A. Then $Cl_{\alpha,X}(A) = A \cup A'_\alpha$.*

Proof. If $x \in A'_\alpha$ then every α-open set containing x contains a point of A other than x. Then by Theorem 4.5(a) we have $x \in Cl_{\alpha,X}(A)$. This implies $A'_\alpha \subseteq Cl_{\alpha,X}(A)$ and thus $A \cup A'_\alpha \subseteq Cl_{\alpha,X}(A)$.

Conversely, let $x \in Cl_{\alpha,X}(A)$. If $x \in A$, then $x \in A \cup A'_\alpha$. Suppose if $x \notin A$, then since $x \in Cl_{\alpha,X}(A)$, every α-open set containing x must intersect A at a point different from x. Thus $x \in A'_\alpha$, so that $x \in A \cup A'_\alpha$ as desired. ∎

Theorem 4.8. *Let $A \subseteq X$ and $\mathcal{C}_X(A) > \alpha$, then A contains all its α-limit points.*

Proof. As $\mathcal{C}_X(A) > \alpha$, by Theorem 4.3, we have $Cl_{\alpha,X}(A) = A$. But by Theorem 4.7 $A \cup A'_\alpha = Cl_{\alpha,X}(A)$. ∎

Definition 4.9. *A co-smooth fuzzy topological space $(X, \mathcal{T}_X)$ is said to satisfy the α-T_1 axiom, if $\mathcal{C}_X(\{x\}) > \alpha$ for all $x \in X$.*

Theorem 4.10. *Let $(X, \mathcal{T}_X)$ be a co-smooth fuzzy topological space satisfying the α-T_1 axiom and let A be a subset of X. Then a point x is an α-limit point of A if and only if every α-open set containing x contains infinitely many points of A.*

Proof. Let $x \in X$ and $A \subseteq X$. If every α-open set containing x contains infinitely many points of A, then by Definition 4.6, it follows that x is an α-limit point of A.

Conversely, suppose that x is an α-limit point of A and let U be an α-open set containing x that contains only finitely many points $\{x_1, x_2, \ldots, x_n\}$ of A other than x. Since X satisfies α-T_1 axiom, we have $\mathcal{C}_X(\{x_i\}) > \alpha$ for all i and hence $\mathcal{C}_X(\{x_1, x_2, \ldots, x_n\}) > \alpha$. Thus it follows that $\mathcal{T}_X(X - \{x_1, x_2, \ldots, x_n\}) > \alpha$. Since $\mathcal{T}_X(U) > \alpha$, we have,

$\mathcal{T}_X(U \cap (X - \{x_1, x_2, \ldots, x_n\})) > \alpha$. Thus $U \cap (X - \{x_1, x_2, \ldots, x_n\})$ is an α-open set containing x that does not contain a point of A other than x. This contradicts the assumption that x is an α-limit point of A. Thus every α-open set containing x contains infinitely many points of A. ∎

We note that when the gradation of openness of a set is α, then the set is not an α-open set according to Definition 2.5; it will be α-open only if its gradation is strictly greater than α. The same problem continues in the cases of α-closed sets, α-interior and α-closure. This seems to be a drawback. So one may think of redefining Definition 2.5 by replacing $\mathcal{T}_X(A) > \alpha$ by $\mathcal{T}_X(A) \geq \alpha$. But if we redefine like this, then the converses of Theorems 4.1, 4.4 and 4.5(b) does not hold (see Examples 4.12, 4.13, 4.14). However in the redefined case also the other parts of Theorems 4.1, 4.4 and 4.5(b) do hold. As they can be proved similar to the proof given to them we skip the proof.

To discuss the above said concepts we give the definition of an α-open set and an α-closed set by replacing $\mathcal{T}_X(A) > \alpha$ by $\mathcal{T}_X(A) \geq \alpha$ as Definition 4.11. We note that the following definition is given just to discuss certain concepts and this is not the definition for α-open and α-closed.

Definition 4.11. *A subset A of a co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to be α-open, $\alpha \in [0,1]$ if $\mathcal{T}_X(A) \geq \alpha$ and is said to be α-closed if the set $X - A$ is α-open. That is, if A is α-closed, then $\mathcal{C}_X(A) \geq \alpha$.*

Example 4.12. *Let $X = \mathbb{R}^2$ and define $\mathcal{T}_X : \mathscr{P}(X) \to [0,1]$ as follows:*

$$\mathcal{T}_X(E) = \begin{cases} 1 & \text{if } E \in \{X, \emptyset\} \\ (1 - \frac{1}{n}) & \text{if } E = (-1,1) \times \left(-\frac{1}{n}, \frac{1}{n}\right),\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

We first prove that $\mathcal{T}_X$ is a topology on X. By the definition of $\mathcal{T}_X$, (i) and (ii) of the Definition 2.1 follows trivially.

Let A and B be any two subsets of X. If any one of the sets A and B is equal to X or $\emptyset$, then (iii) of Definition 2.1 follows trivially. Suppose one of the sets A and B does not belong to the collection $\{X, \emptyset, \{(-1,1) \times \left(-\frac{1}{n}, \frac{1}{n}\right)\}_{n \in \mathbb{N}}\}$, then $\mathcal{T}_X(A) \wedge \mathcal{T}_X(B) = 0 \leq \mathcal{T}_X(A \cap B)$. Now suppose $A = (-1,1) \times \left(-\frac{1}{n}, \frac{1}{n}\right)$ and $B = (-1,1) \times \left(-\frac{1}{m}, \frac{1}{m}\right)$ for some $n \leq m$. Then clearly $A \cap B = B$ and hence $\mathcal{T}_X(A \wedge B) \geq \mathcal{T}_X(A) \wedge \mathcal{T}_X(B)$ follows.

Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be any collection of subsets of X. If $\underset{\lambda \in \Lambda}{\cup} A_\lambda$ is X or $\emptyset$, then (iv) of Definition 2.1 follows. If $A_{\lambda_0} \notin \{(-1,1) \times \left(-\frac{1}{n}, \frac{1}{n}\right)\}_{n \in \mathbb{N}}$ for some λ_0, then $\mathcal{T}_X(A_{\lambda_0}) = 0$ and hence (iv) of Definition 2.1 follows in this case. So let $A_\lambda = (-1,1) \times \left(-\frac{1}{n_\lambda}, \frac{1}{n_\lambda}\right)$ for all $\lambda \in \Lambda$, where $n_\lambda \in \mathbb{N}$. Let $n_{\lambda_0} = \inf n_\lambda$, then we have $\frac{1}{n_{\lambda_0}} \geq \frac{1}{n_\lambda}$ for all λ and hence $\underset{\lambda \in \Lambda}{\cap} A_\lambda = (-1,1) \times \left(-\frac{1}{n_{\lambda_0}}, \frac{1}{n_{\lambda_0}}\right)$. Let $A_{\lambda_0} = (-1,1) \times \left(-\frac{1}{n_{\lambda_0}}, \frac{1}{n_{\lambda_0}}\right)$, then, $\mathcal{T}_X(\underset{\lambda \in \Lambda}{\cap} A_\lambda) = \mathcal{T}_X(A_{\lambda_0}) \geq \underset{\lambda \in \Lambda}{\wedge} \mathcal{T}_X(A_\lambda)$. Thus (iv) of Definition 2.1 follows in all the cases. Thus $\mathcal{T}_X$ is a co-smooth fuzzy topology on X.

Now let $Y = \mathbb{R} \times \{0\}$ and let $\mathcal{T}_Y$ be the subspace co-smooth fuzzy topology on Y. Let $A' = (-1,1) \times \{0\}$ and let $B_n = (-1,1) \times \left(-\frac{1}{n}, \frac{1}{n}\right)$ $n \in \mathbb{N}$. Then we have $B_n \cap Y = A'$ and $\mathcal{T}_X(B_n) = 1 - \frac{1}{n}$, for all $n \in \mathbb{N}$. Thus it follows that,

$$\mathcal{T}_Y(A') = \sup\{\mathcal{T}_X(B)/B \cap Y = A', B \subseteq X\} \geq \sup\{\mathcal{T}_X(B_n)\} = 1$$

and hence $\mathcal{T}_{Y}(A') = 1$. *Now let* $A = ((-\infty, -1] \cup [1, \infty)) \times \{0\}$. *Then*

$$\mathcal{C}_{Y}(A) = \mathcal{T}_{Y}((-1,1) \times \{0\}) = \mathcal{T}_{Y}(A') = 1,$$

But there exists no $B \subseteq X$ *such that* $B \cap Y = A$ *and* $\mathcal{C}_{X}(B) = 1$. *Thus the converse of the Theorem 4.1 is not true if we refine Definition 2.5 by replacing* $\mathcal{T}_{X}(A) > \alpha$ *by* $\mathcal{T}_{X}(A) \geq \alpha$.

Example 4.13. *Let* $X = \mathbb{R}^2$, $Y = \mathbb{R} \times \{0\}$ *and* $\mathcal{T}_{X} : \mathscr{P}(X) \to [0,1]$ *be*

$$\mathcal{T}_{X}(E) = \begin{cases} 1 & \text{if } E \in \{X, \emptyset\} \\ \frac{1}{2}(1 - \frac{1}{n}) & \text{if } E = (-\infty, 0) \times \left(-\frac{1}{n}, \frac{1}{n}\right), \ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{T}_{X}$ *is a topology on* X.

Let $A' = (-\infty, 0) \times \{0\}$ *and* $\mathcal{T}_{Y}$ *be the subspace co-smooth fuzzy topology on* Y. *Let* $B_n = (-\infty, 0) \times \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, *then* $B_n \cap Y = A'$ *and* $\mathcal{T}_{X}(B_n) = \frac{1}{2}(1 - \frac{1}{n})$, *for all* $n \in \mathbb{N}$. *Thus*

$$\mathcal{T}_{Y}(A') = \sup\{\mathcal{T}_{X}(B) / B \cap Y = A', B \subseteq X\} \geq \sup\{\mathcal{T}_{X}(B_n)\} = \frac{1}{2}.$$

Now let $A = [0, \infty) \times \{0\}$, *then* $\mathcal{C}_{Y}(A) = \mathcal{T}_{Y}((-\infty, 0) \times \{0\}) \geq \frac{1}{2}$. *Thus* A *is an* α*-closed set, according to Definition 4.11. Thus by the definition of* α*-closure of a set, we have* $Cl_{\frac{1}{2},Y}(A) = A$. *But since there exists no* $E \subseteq X$ *such that* $A \subseteq E$ *and* $\mathcal{C}_{X}(E) \geq \frac{1}{2}$ *other than* X, *we have* $Cl_{\frac{1}{2},X}(A) = X$. *This implies that* $Cl_{\frac{1}{2},X}(A) \cap Y = Y$ *and hence* $Cl_{\frac{1}{2},X}(A) \cap Y \nsubseteq Cl_{\frac{1}{2},Y}(A)$. *The reverse inequality in Theorem 4.4 is not true if we refine Definition 2.5 by replacing* $\mathcal{T}_{X}(A) > \alpha$ *by* $\mathcal{T}_{X}(A) \geq \alpha$.

Example 4.14. *Let* $X = (0,1)$ *and define* $\mathcal{B} : \mathscr{P}(X) \to [0,1]$ *as follows:*

$$\mathcal{B}(E) = \begin{cases} 1 & \text{if } E = X \text{ or } \emptyset \\ 1 - a & \text{if } E = (a, 1) \text{ where } a < \frac{1}{2} \\ b - \frac{1}{2} & \text{if } E = [\frac{1}{2}, b] \text{ where } b < 1 \\ 0 & \text{otherwise} \end{cases}$$

We first claim that $\mathcal{B}$ *is a basis for a co-smooth fuzzy topology on* X.

As $\mathcal{B}(X) = 1$, *(i) of Definition 2.2 follows. Now we prove (ii) of Definition 2.2. Let* $x \in A \cap B$ *and* $\epsilon > 0$. *If both the sets* A *and* B *are of the form* $(a, 1)$ *or of the form* $[\frac{1}{2}, b]$. *Then by taking* $C = A \cap B$, *(ii) of Definition 2.2 follows. If* $A = (a, 1)$ *for some* $a < \frac{1}{2}$ *and* $B = [\frac{1}{2}, b]$ *for some* $b < 1$, *then* $A \cap B = B$. *Thus by taking* $C = A \cap B$, *(ii) of Definition 2.2 follows. Hence* $\mathcal{B}$ *is a basis.*

Let $\mathcal{T}_{X}$ *be the co-smooth fuzzy topology generated by* $\mathcal{B}$. *Let* $A = (0, \frac{1}{2})$ *and* $x = \frac{1}{2}$, *then clearly* $x \notin A$. *Now we claim that every* $B \subseteq X$ *containing* x, *with* $\mathcal{B}(B) > \frac{1}{2}$ *intersects* A.

By the construction of $\mathcal{B}$ *the sets of the form* $B = (a, 1)$, $a < \frac{1}{2}$ *are the only sets containing* $x = \frac{1}{2}$ *and* $\mathcal{B}(B) > \frac{1}{2}$. *Clearly all them intersects* A. *Let* $\epsilon > 0$, *choose* $N > 2$ *such that* $\frac{1}{N} < \epsilon$. *Let* $A_n = [\frac{1}{2}, 1 - \frac{1}{n}]$, $n \geq N$. *Then* $\bigcup_{n=N}^{\infty} A_n = [\frac{1}{2}, 1)$. *Now by the definition of* $\mathcal{B}$, *we have*

$$\mathcal{B}(A_n) = \left(1 - \frac{1}{n}\right) - \frac{1}{2} = \frac{1}{2} - \frac{1}{n}.$$

But since $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. Thus it follows that $\frac{1}{2} - \frac{1}{n} > \frac{1}{2} - \epsilon$ and hence $\mathcal{B}(A_n) > \frac{1}{2} - \epsilon$, for all $n \geq N$. Thus for every $\epsilon > 0$, there exists an inner cover $\{A_n\}$ for $[\frac{1}{2}, 1)$ with

$$\inf\{\mathcal{B}(A_n)\} > \frac{1}{2} - \epsilon.$$

This implies that $\mathcal{T}_X([\frac{1}{2}, 1)) \geq \frac{1}{2}$. Thus we have,

$$\mathcal{C}_X(A) = \mathcal{T}_X(X - A) = \mathcal{T}_X\left(X - \left(0, \frac{1}{2}\right)\right) = \mathcal{T}_X\left([\frac{1}{2}, 1)\right) \geq \frac{1}{2}$$

which implies, $Cl_{\frac{1}{2},X}(A) = A$ and hence $\frac{1}{2} \notin Cl_{\frac{1}{2},X}(A)$. Thus there exist $x \in X$ and $A \subseteq X$ such that every $B \subseteq X$ containing x, with $\mathcal{B}(B) > \frac{1}{2}$ intersects A, but $x \notin Cl_{\frac{1}{2},X}(A)$. Thus the converse of the Theorem 4.5(b) is not true if we refine Definition 2.5 by replacing $\mathcal{T}_X(A) > \alpha$ by $\mathcal{T}_X(A) \geq \alpha$.

5. α-Compactness

The compactness of a co-smooth fuzzy topology is defined and discussed in [11] as a function from a class of co-smooth fuzzy topological spaces to $[0, 1]$. But we define compactness as a function from $\mathscr{P}(X)$ to $[0, 1]$ at a level $\alpha \in [0, 1)$. we also define the index of compactness of a set and prove that the product of finitely many α-compact spaces is α-compact.

Definition 5.1. *A collection $\mathscr{A}$ of subsets of a co-smooth fuzzy topological space $(X, \mathcal{T}_X)$ is said to cover X, if $X = \bigcup_{A \in \mathscr{A}} A$. It is called an α-open $(\alpha \in [0, 1))$ covering of X if its elements are α-open subsets of X.*

Let Y be a subspace of X, a collection $\mathscr{A}$ of subsets of X is said to cover Y if the union of its elements contains Y.

Definition 5.2. *A co-smooth fuzzy topological space $(X, \mathcal{T})$ is said to be α-compact if every α-open covering of X contains a finite subcollection covering X. A subset A of X is said to be α-compact if $(A, \mathcal{T}_A)$ is α-compact. The index of compactness of a subset A of X is defined as $1 - \inf\{\alpha / A$ is α-compact$\}$.*

If X is a crisp compact topological space, then it can be viewed as a co-smooth fuzzy topological space by defining $\mathcal{T}(A) = 1$ if A is open and $\mathcal{T}(A) = 0$ if A is not open. It easy to see that the index of compactness of X is one. We now give a nontrivial example.

Example 5.3. *Let $X = \mathbb{R}$, define $\mathcal{T} : \mathscr{P}(X) \to [0, 1]$ as follows:*

$$\mathcal{T}(E) = \begin{cases} 1 & \text{if } E \in \{X, \emptyset\} \\ \frac{3}{4} & \text{if } E = (a, b), \text{ where } a, b \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

then $\mathcal{T}$ is a co-smooth fuzzy topology on X and X is α-compact for all $\alpha > \frac{3}{4}$. Thus the index of compactness of X with respect to $\mathcal{T}$ is $\frac{1}{4}$.

Theorem 5.4. *Let Y be a subspace of a co-smooth fuzzy topological space $(X, \mathcal{T}_X)$ with subspace topology $\mathcal{T}_Y$. Then Y is α-compact if and only if every covering of Y by sets α-open in X contains a finite subcollection covering Y.*

Proof. Suppose that Y is α-compact and $\mathscr{A} = \{A_\lambda\}_{\lambda\in\Lambda}$ is a covering of Y by sets α-open in X. Then the collection $\{A_\lambda \cap Y/\lambda \in \Lambda\}$ is a covering of Y by sets α-open in Y, hence there exists a finite subcollection $\{A_{\lambda_1} \cap Y, A_{\lambda_2} \cap Y, \ldots, A_{\lambda_n} \cap Y\}$ that covers Y. Then $\{A_{\lambda_1}, A_{\lambda_2}, \ldots, A_{\lambda_n}\}$ is a subcollection of $\mathscr{A}$ that covers Y.

Conversely, suppose every covering of Y by sets α-open in X contains a finite subcollection covering Y; we claim that Y is α-compact. Let $\mathscr{A}' = \{A'_\lambda\}_{\lambda\in\Lambda}$ be a covering of Y by sets α-open in Y. Let $\mathcal{T}_Y(A'_\lambda) = \gamma_\lambda$. For each λ, choose $\epsilon_\lambda > 0$ such that $\alpha < \gamma_\lambda - \epsilon_\lambda$. Then for each set A'_λ, there exists a set $A_\lambda \subseteq X$ such that $A_\lambda \cap Y = A'_\lambda$ and $\mathcal{T}_X(A_\lambda) \geq \gamma_\lambda - \epsilon_\lambda > \alpha$. Thus $\{A_\lambda\}_{\lambda\in\Lambda}$ is a covering of Y by sets α-open in X. But by hypothesis, there exists a finite collection $\{A_{\lambda_1}, A_{\lambda_2}, \ldots, A_{\lambda_n}\}$ that covers Y. Then $\{A'_{\lambda_1}, A'_{\lambda_2}, \ldots, A'_{\lambda_n}\}$ is a sub collection of $\mathscr{A}'$ that covers Y. Thus Y is α-compact. ∎

Definition 5.5. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two co-smooth fuzzy topological spaces. A function $f : X \to Y$ is said to be α-continuous if the inverse image of every α-open set in Y is α-open in X. Further if $f : X \to Y$ is a bijection and both the functions f and $f^{-1} : Y \to X$ are α-continuous, then f is called an α-homeomorphism.*

It is easy to prove that a function $f : X \to Y$ is α-continuous if the inverse image of every α-closed set in Y is α-closed in X. Now we state some results whose proofs run parallel to those in the classical topological theory.

i. Every α-closed subspace of an α-compact space is α-compact.
ii. The α-continuous image of an α-compact space is α-compact.

Now we prove a lemma which is similar to the tubes lemma in the classical topology, then we prove that the product of finitely many α-compact spaces is α-compact.

Lemma 5.6. *Let $X \times Y$ be the product space, with product topology $\mathcal{T}_{X\times Y}$, where Y is α-compact. If N is an α-open subset of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a α-open subset of X containing x_0.*

Proof. As N is α-open in $X \times Y$, we have $\mathcal{T}_{X\times Y}(N) > \alpha$. Let $\mathcal{T}_{X\times Y}(N) = \gamma$ and choose $\epsilon > 0$ such that $\gamma - \epsilon > \alpha$. Let $\mathcal{B}_{X\times Y}$ be a basis for the product space, then there exists a collection $\{U_\lambda \times V_\lambda\}_{\lambda\in\Lambda}$ where $U_\lambda \subseteq X$ and $V_\lambda \subseteq Y$ such that

$$N = \bigcup_{\lambda\in\Lambda} U_\lambda \times V_\lambda \quad \text{and} \quad \inf\{\mathcal{B}_{X\times Y}(U_\lambda \times V_\lambda)\} \geq \gamma - \epsilon > \alpha.$$

This implies that $\mathcal{T}_{X\times Y}(U_\lambda \times V_\lambda) > \alpha$ for all $\lambda \in \Lambda$. Then clearly the collection $\{U_\lambda \times V_\lambda\}_{\lambda\in\Lambda}$ is a cover for the slice $x_0 \times Y$ by sets α-open in $X \times Y$. And since $x_0 \times Y$ is α-compact being α-homeomorphic to Y. Therefore we can cover $x_0 \times Y$ by finitely many such elements

$$U_{\lambda_1} \times V_{\lambda_1}, U_{\lambda_2} \times V_{\lambda_2}, \ldots, U_{\lambda_n} \times V_{\lambda_n}.$$

Define $W = U_{\lambda_1} \cap \cdots \cap U_{\lambda_n}$, then the set W is α-open in X and clearly it contains x_0.

Now we claim that $W \times Y \subseteq (U_{\lambda_1} \times V_{\lambda_1}) \cup (U_{\lambda_2} \times V_{\lambda_2}) \cup \cdots \cup (U_{\lambda_n} \times V_{\lambda_n})$. For, let $x \times y$ be a point of $W \times Y$. Consider $x_0 \times y$ of the slice $x_0 \times Y$ having the same y-coordinate as this point. Now $x_0 \times y$ belongs to $U_{\lambda_i} \times V_{\lambda_i}$ for some i, so that $y \in V_{\lambda_i}$. But since $x \in W$ we have $x \in U_j$ for every j. Therefore, we have $x \times y \in U_{\lambda_i} \times V_{\lambda_i}$, as desired.

Since all the sets $U_{\lambda_i} \times V_{\lambda_i}$ lie in N and since they cover $W \times Y$, the tube $W \times Y$ also contained in N. ∎

Theorem 5.7. *Let X be an α-compact set and let Y be a β-compact set. Let $\gamma = \max\{\alpha, \beta\}$, then the product space $X \times Y$ is γ-compact.*

Proof. Without loss of generality let us assume that $\alpha \geq \beta$. We claim that $X \times Y$ is α-compact. Let $\mathscr{A} = \{A_\lambda\}$ be an α-open covering of $X \times Y$. This implies that the openness of each A_λ is strictly greater than α and also strictly greater than β. Thus the collection can be considered as a β-open covering of $X \times Y$. Let $x_0 \in X$. Then since the slice $x_0 \times Y$ is β-compact being β-homeomorphic to Y, it can be covered by finitely many elements $A_1, A_2, \ldots, A_n$ of $\mathscr{A}$. Let $N = A_1 \cup \cdots \cup A_n$, then N is an α-open set containing the slice $x_0 \times Y$. But by Lemma 5.6, we can find a tube $W \times Y \subseteq N$ about $x_0 \times Y$, where W is α-open in X. Then $W \times Y$ can be covered by finitely many elements $A_1, A_2, \ldots, A_n$ of $\mathscr{A}$. Thus for each $x \in X$, we can find an α-open set W_x of x such that the tube $W_x \times Y$ can be covered by finitely many elements of $\mathscr{A}$. Clearly, the collection of all such α-open sets W_x is covering of X. Now, since X is α-compact, there exists a finite subcollection

$$\{W_1, \ldots, W_k\}$$

covering X. Then the union of the tubes

$$W_1 \times Y, W_2 \times Y, \ldots, W_k \times Y$$

is equal to $X \times Y$, since each tube can be covered by finitely many elements of $\mathscr{A}$, the space $X \times Y$ can be covered by finitely many elements of $\mathscr{A}$. ∎

Corollary 5.8. *The product of finitely many α-compact spaces is α-compact.*

Corollary 5.9. *The index of compactness of the product of finitely many co-smooth fuzzy topological spaces is the minimum of the indices of compactness of the spaces.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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