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HIGHER ORDER DERIVATIVE-FREE ITERATIVE METHODS WITH AND WITHOUT MEMORY IN BANACH SPACE UNDER WEAK CONDITIONS

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Abstract We study the method considered in Ahmad et al. (2016), for solving systems of nonlinear equations, modified suitably to include the nonlinear equations in Banach spaces. We use the idea of restricted convergence domains instead of Taylors expansion in our convergence analysis and our conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

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1. Introduction

In [8], Ahmad et al. considered the following method

$$\begin{aligned} y_n &= x_n - q_1, \\ z_n &= y_n - a_0 q_2 - (3-2a_0)q_3 - (a_0-2)q_4, \\ x_{n+1} &= z_n - a_1 q_5 - a_2 q_6 - a_3 q_7 - a_4 q_8 - a_5 q_9; \end{aligned} \tag{1.1}$$

$u_n = x_n + b_0 F(x_n)$, $A_n = [x_n, u_n, F]$, $h_n = y_n + b_1 F(y_n)$, $B_n = [h_n, y_n : F]$, $l_n = z_n + b_2 F(z_n)$, $C_n = [l_n, z_n; F], b_0, b_1, b_2 \in \mathbb{R}$, $A_n q_1 = F(x_n), A_n q_2 = F(y_n), A_n q_3 =$

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$B_n q_2, A_n q_4 = B_n q_3, A_n q_5 = F(z_n),\ A_n q_6 = C_n q_5, A_n q_7 = C_n q_6, A_n q_8 = C_n q_7, A_n q_9 = C_n q_8,\ a_1 = 4 + a_5, a_2 = -6 - 4a_5, a_3 = 4 + 6a_5, a_4 = -1 - 4a_5$. and $[.,.;F] : D^2 \longrightarrow L(X)$ is a divided difference of order one on D^2 [2], for solving systems of equations, provided that $X = \mathbb{R}^j$ and $D \subset X$. In this study we consider method (1.1), but for approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.2}$$

where F is a continuous operator defined on a convex subset D of a Banach space X with values in X. The divided difference of order one satisfies $[x, y; F](x-y) = F(x) - F(y)$ for $x \neq y$ with $x, y \in D$ and $[x, x; F] = F'(x)$, $x \in D$, if F is Fréchet-differentiable on D. We use the notation $U(u, \rho)$, $\overline{U}(u, \rho)$ to denote the open and closed balls in X, respectively with center $u \in X$ and of radius $\rho > 0$.

Finding solution for (1.2) is an important problem in mathematics due to the wide application of the equation (1.2). Convergence analysis of the method in [2] used assumptions on the eighth Fréchet-derivative although no derivative appears in the method. This assumption on the higher order Fréchet derivatives of the operator F restricts the applicability of method (1.1). For example consider the following;

EXAMPLE 1.1. Let $X = C[0,1], D = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [4, 7, 20] defined by

$$x(s) = \int_0^1 G(s,t)\frac{x(t)^2}{2}dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.2), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\frac{x(t)^2}{2}dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)x(t)dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\|x - y\|.$$

One can see that the work in [2] cannot be applied in this setting, if we choose $[x, y; F] = \int_0^1 F'(y + \theta(x-y))d\theta$.

Our goal is to weaken the assumptions in [2] and apply the method for solving equation (1.2) in Banach spaces, so that the applicability of the method (1.1) can be extended. The technique of restricted convergence domains can apply on other iterative methods [1, 3–25].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in [2]. Special cases and numerical examples are given in the last section.

2. Local convergence analysis

Let $b_0, b_1, b_2 \in \mathbb{R}$. Let $w_0 : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$, $v_0 : [0,+\infty) \longrightarrow [0,+\infty)$ be a continuous, non-decreasing in both variables satisfying $w_0(0,0) = 0$. Define the parameter r_0 by

$$r_0 = \sup\{t \geq 0 : w_0(t, (1 + |b_0|v_0(t))t) < 1\}. \tag{2.1}$$

Let also $v, w : [0, r_0) \times [0, r_0) \to [0,+\infty)$ be continuous and non-decreasing functions with $w(0,0) = 0$. Define functions g_i, $\bar{g}_i$, $i = 1, 2, 3$ on the interval $[0, r_0)$ by

$$g_1(t) = \frac{w(t, t + |b_0|v_0(t)t)}{1 - p(t)},\ p(t) = w_0(t, t + |b_0|v_0(t)t),$$

$$\bar{g}_1(t) = g_1(t) - 1,$$

$$\begin{aligned} g_2(t) &= \Bigg[\frac{w(t + g_1(t)t, t + |b_0|v_0(t)t)}{1 - p(t)} + \frac{|1 - a_0|v(g_1(t)t, 0)}{1 - p(t)} \\ &\quad + |3 - 2a_0|\frac{v(g_1(t)t + |b_1|v_0(g_1(t)t)g_1(t)t, g_1(t)tv(g_1(t)t, 0)}{(1 - p(t))^2} \\ &\quad + |a_0 - 2|\frac{v(g_1(t)t + |b_1|v_0(g_1(t)t)g_1(t)t, g_1(t)t^2v(g_1(t)t, 0)}{(1 - p(t))^3}\Bigg] g_1(t), \end{aligned}$$

$$\bar{g}_2(t) = g_2(t) - 1,$$

$$\begin{aligned} g_3(t) &= \Bigg[\frac{w(t + g_2(t)t, t + |b_0|v_0(t)t)}{1 - p(t)} + \frac{|1 - a_1|v(g_2(t)t, 0)}{1 - p(t)} \\ &\quad + |a_2|\frac{v(g_2(t)t + |b_2|v_0(g_2(t)t)g_2(t)t, g_2(t)t)}{(1 - p(t))^2} \\ &\quad + |a_3|\frac{v(g_2(t)t + |b_2|v_0(g_2(t)t)g_2(t)t, g_2(t)t)^2}{(1 - p(t))^3} \\ &\quad + |a_4|\frac{v(g_2(t)t + |b_2|v_0(g_2(t)t)g_2(t)t, g_2(t)t)^3}{(1 - p(t))^4} \\ &\quad + |a_5|\frac{v(g_2(t)t + |b_2|v_0(g_2(t)t)g_2(t)t, g_2(t)t)^4}{(1 - p(t))^5}\Bigg] g_2(t), \end{aligned}$$

and

$$\bar{g}_3(t) = g_3(t) - 1.$$

We have that $\bar{g}_i(0) = -1 < 0$ and $\bar{g}_i(t) \to +\infty$ as $t \to r_0^-$. By applying the intermediate value theorem on the functions $\bar{g}_i$ defined on the interval $[0, r_0]$, we deduce that there exist zeros on the interval $(0, r_0)$. Denote by r_i the smallest zero of functions $\bar{g}_i$ on the interval $(0, r_0)$.. Define the radius of convergence r by

$$r = min\{r_i\}, i = 1, 2, 3. \tag{2.2}$$

Then, for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \tag{2.3}$$

and

$$0 \leq p(t) < 1. \tag{2.4}$$

Define also parameter R by

$$R = \max\{r, r+|b_0|v_0(r)r, g_1(r)r+|b_1|v_0(g_1(r)r)g_1(r)r, g_2(r)r+|b_2|v_0(g_2(r)r)g_2(r)r\}. \tag{2.5}$$

Next, we present the local convergence analysis of method (1.1) under the preceding notation.

THEOREM 2.1. Let $F : D \subset X \to X$ be a continuous operator. Let also $[.,.;F] : D \times D \to L(X)$ be a divided difference of order one. Suppose:
there exists $x^* \in D$ such that operator F is Fréchet-differentiable at $x = x^*$,

$$F(x^*) = 0, \ F'(x^*)^{-1} \in L(X); \tag{2.6}$$

there exist functions $w_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, $v_0 : [0, +\infty) \to [0, +\infty)$ continuous, nondecreasing with $w_0(0, 0) = 0$ such that for each $x, y \in D$,

$$\|F'(x^*)^{-1}([y, x; F] - F'(x^*))\| \leq w_0(\|y - x^*\|, \|x - x^*\|), \tag{2.7}$$

$$\|[x, x^*; F]\| \leq v_0(\|x - x^*\|); \tag{2.8}$$

there exist functions $w : [0, r_0) \times [0, r_0) \to [0, +\infty), v : [0, r_0) \times [0, r_0) \to [0, +\infty)$ continuous, non-decreasing with $w(0, 0) = 0$ such that for each $x, y \in D_0 = D \cap U(x_0, r_0)$

$$\|F'(x^*)^{-1}([x, y; F] - [x, x^*; F])\| \leq w(\|x - x^*\|, \|y - x^*\|) \tag{2.9}$$

$$\|F'(x^*)^{-1}[x, y; F]\| \leq v(\|x - x^*\|, \|y - x^*\|) \tag{2.10}$$

and

$$\overline{U}(x^*, R) \subseteq D \tag{2.11}$$

where r_0, r and R are given by (2.1), (2.2) and (2.5) respectively. Then sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.1) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.12}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.13}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.14}$$

where, the functions $g_i, i = 1, 2, 3$ are defined previously. Furthermore, if there exists for $R^* \geq r$, such that

$$w_0(R^*, 0) < 1 \ \ or \ \ w_0(0, R^*) < 1, \tag{2.15}$$

then, the limit point x^* is the only solution of equation $F(x) = 0$ in $D_1 := D \cap \overline{U}(x^*, R^*)$.

Proof. We shall show estimates (2.12)-(2.14) using induction on $k = 0, 1, 2\cdots$. Using (2.6) and (2.8), we have that

$$\begin{aligned} \|u_0 - x^*\| &= \|x_0 - x^* + b_0(F(x_0) - F(x^*))\| \\ &\leq \|x_0 - x^* + b_0[x_0, x^*; F](x_0 - x^*)\| \\ &\leq \|x_0 - x^*\| + |b_0|v_0(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq r + |b_0|v_0(r)r \leq R, \end{aligned} \tag{2.16}$$

so $u_0 \in \bar{U}(x^*, R)$. Then, by (2.2), (2.4), (2.5), (2.6), (2.7) and (2.16), we get that

$$\begin{aligned} \|F'(x^*)^{-1}([x_0, u_0; F] - F'(x^*))\| &\leq w_0(\|x_0 - x^*\|, \|u_0 - x^*\|) \\ &\leq w_0(R, R + |b_0|v_0(R)R) \\ &= p(R) < 1. \end{aligned} \tag{2.17}$$

It follows from (2.17) and the Banach Lemma on invertible operators [4–7] that $A_0 = [x_0, u_0; F]^{-1} \in L(X)$ and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|, \|u_0 - x^*\|)}. \tag{2.18}$$

We also have that y_0, z_0, x_1 are well defined by the method (1.1) for $n = 0$. By (2.2), (2.3)(for $i = 1$), (2.6), (2.9), (2.18) and the first substep of method (1.1) we obtain in turn that

$$\begin{aligned} \|y_0 - x^*\| &= \|x_0 - x^* - A_0^{-1}F'(x_0)\| \\ &\leq \|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0(x_0 - x^*) - F(x_0))\| \\ &\leq \|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0 - [x_0, x^*; F])(x_0 - x^*)\| \\ &\leq \frac{w(\|x_0 - x^*\|, \|x_0 - x^*\| + |b_0|v_0(\|x_0 - x^*\|)\|x_0 - x^*\|)\|x_0 - x^*\|}{1 - p(\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.19}$$

which shows (2.12) for $k = 0$ and $y_0 \in U(x^*, r)$. As in (2.19) (with $y_0 = x_0$), we obtain the estimates

$$\begin{aligned} &\|y_0 - x^* - A_0^{-1}F(y_0)\| \\ \leq\ & \|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0 - [y_0, x^*; F])(y_0 - x^*)\| \\ \leq\ & \frac{w(\|x_0 - x^* + x^* - y_0\|, \|u_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)}\|y_0 - x^*\| \\ \leq\ & \frac{w(\|x_0 - x^*\| + \|y_0 - x^*\|, \|x_0 - x^*\| + |b_0|v_0(\|x_0 - x^*\|)\|x_0 - x^*\|)\|y_0 - x^*\|}{1 - p(\|x_0 - x^*\|)} \end{aligned} \tag{2.20}$$

and

$$\|(1 - a_0)A_0^{-1}F(y_0)\| \leq \frac{|1 - a_0|v(\|y_0 - x^*\|, 0)\|y_0 - x^*\|}{1 - p(\|x_0 - x^*\|)}. \tag{2.21}$$

In view of the second substep of method (1.1) for $n = 0$, (2.2), (2.3) (for $i = 2$), (2.10), (2.19)-(2.21) we get in turn that

$$\begin{aligned}\|z_0 - x^*\| &\leq \|y_0 - x^* - A_0^{-1}F(y_0)\| + \|(1-a_0)A_0^{-1}F(y_0)\| + \|(3-2a_0)A_0^{-1}B_0A_0^{-1}F(y_0)\| \\ &\quad + \|(a_0-2)A_0^{-1}B_0A_0^{-1}F(y_0)\| \\ &\leq \left[\frac{w(\|x_0 - x^*\| + \|y_0 - x^*\|, \|x_0 - x^*\| + |b_0|v_0(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - p(\|x_0 - x^*\|)}\right. \\ &\quad + \frac{|1-a_0|v(\|y_0 - x^*\|, 0)}{1 - p(\|x_0 - x^*\|)} \\ &\quad + |3 - 2a_0|\frac{v(\|h_0 - x^*\|, \|y_0 - x^*\|)v(\|y_0 - x^*\|, 0)}{(1 - p(\|x_0 - x^*\|))^2} \\ &\quad \left. + |a_0 - 2|\frac{v(\|h_0 - x^*\|, \|y_0 - x^*\|)^2 v(\|y_0 - x^*\|, 0)}{(1 - p(\|x_0 - x^*\|))^3}\right] \|y_0 - x^*\| \\ &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.22}$$

which shows (2.13) and $z_0 \in U(x^*, r)$. We also used

$$\begin{aligned}\|h_0 - x^*\| &\leq \|y_0 - x^*\| + |b_1|v_0(\|y_0 - x^*\|)\|y_0 - x^*\| \\ &\leq g_1(r)r + |b_1|v_0(g_1(r)r)g_1(r)r \leq R.\end{aligned} \tag{2.23}$$

As in (2.20), (2.21) (for $y_0 = z_0$), we obtain the estimates

$$\begin{aligned}&\|z_0 - x^* - A_0^{-1}F(z_0)\| \\ \leq\ &\frac{w(\|x_0 - x^*\| + \|z_0 - x^*\|, \|x_0 - x^*\| + |b_0|v_0(\|x_0 - x^*\|, \|x_0 - x^*\|)\|z_0 - x^*\|}{1 - p(\|x_0 - x^*\|)}\end{aligned} \tag{2.24}$$

and

$$\|(1 - a_1)A_0^{-1}F(z_0)\| \leq \frac{|1 - a_1|v(\|z_0 - x^*\|, 0)\|z_0 - x^*\|}{1 - p(\|x_0 - x^*\|)}. \tag{2.25}$$

Then, using the third substep of method (1.1) for $n = 0$, (2.2), (2.3) (for $i = 3$), (2.18), (2.19) (2.22), (2.24) and (2.25), we have in turn that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|z_0 - x^* - A_0^{-1}F(z_0)\| + \|(1-a_1)A_0^{-1}F(z_0)\| \\ &+\|a_2A_0^{-1}C_0A_0^{-1}F(z_0)\| + \|a_3A_0^{-1}C_0A_0^{-1}C_0A_0^{-1}F(z_0)\| \\ &+\|a_4A_0^{-1}C_0A_0^{-1}C_0A_0^{-1}C_0A_0^{-1}(z_0)\| + \|a_5A_0^{-1}C_0A_0^{-1}C_0A_0^{-1}C_0A_0^{-1}C_0A_0^{-1}F(z_0)\| \\ &\leq \Bigg[\frac{w(\|x_0-x^*\|+\|z_0-x^*\|, \|x_0-x^*\|+|b_0|v_0(\|x_0-x^*\|)\|x_0-x^*\|)}{1-p(\|x_0-x^*\|)} \\ &+\frac{|1-a_1|v(\|z_0-x^*\|,0)}{1-p(\|x_0-x^*\|)} \\ &+|a_2|\frac{v(\|z_0-x^*\|+|b_2|v_0(\|z_0-x^*\|)\|z_0-x^*\|, \|z_0-x^*\|)}{(1-p(\|x_0-x^*\|))^2} \\ &+|a_3|\frac{v(\|z_0-x^*\|+|b_2|v_0(\|z_0-x^*\|)\|z_0-x^*\|, \|z_0-x^*\|)^2}{(1-p(\|x_0-x^*\|))^3} \\ &+|a_4|\frac{v(\|z_0-x^*\|+|b_2|v_0(\|z_0-x^*\|)\|z_0-x^*\|, \|z_0-x^*\|)^3}{(1-p(\|x_0-x^*\|))^4} \\ &+|a_5|\frac{v(\|z_0-x^*\|+|b_2|v_0(\|z_0-x^*\|)\|z_0-x^*\|, \|z_0-x^*\|)^4}{(1-p(\|x_0-x^*\|))^5}\Bigg] \\ &\leq g_3(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \end{aligned} \tag{2.26}$$

which shows (2.14) for $n = 0$ and $x_1 \in U(x^*, r)$. We also used that

$$\begin{aligned} \|l_0 - x^*\| &\leq \|z_0 - x^*\| + |b_2|v_0(\|z_0-x^*\|)\|z_0-x^*\| \\ &\leq g_2(r)r + |b_2|v_0(g_2(r)r)g_2(r)r \leq R. \end{aligned} \tag{2.27}$$

The induction for error bounds (2.12)-(2.14) is completed in an analogous way, if we replace x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates. Then, in view of the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r, \tag{2.28}$$

here $c = g_3(\|x_0 - x*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally, to show the uniqueness of the solution x^* in D_1, suppose $y^* \in D_1$ with $F(y^*) = 0$. Define the linear operator $Q = [y^*, x^*; F]$. Using (2.7) and (2.15) we obtain that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq w_0(\|y^* - x^*\|, 0) \leq w_0(R^*, 0) < 1, \tag{2.29}$$

so $Q^{-1} \in L(X)$. Then, using the identity

$$0 = F(y^*) - F(x^*) = Q(y^* - x^*), \tag{2.30}$$

we conclude that $x^* = y^*$.

□

REMARK 2.2. (a) A stronger condition depending on function w_0 can replace condition (2.8). Indeed, using (2.7), we have in turn that

$$\begin{aligned} \|[x, x^*; F]\| &= \|F'(x^*)F'(x^*)^{-1}[x, x^*; F]\| \\ &\leq \|F'(x^*)\|[\|F'(x^*)^{-1}([x, x^*; F] - F'(x^*)) + I\|] \\ &\leq \beta(1 + \|F'(x^*)^{-1}([x, x^*; F] - F'(x^*))\|) \\ &\leq \beta(1 + w_0(t, 0)). \end{aligned}$$

Therefore, $v_0(t) := \beta(1 + w_0(t, 0))$ or $v_0(t) := \beta(1 + w_0(r_0, 0))$ or $v_0(t) := \beta(1 + w_0(r_0, 0))$, where $\|F'(x^*)\| \leq \beta$.

(b) It is worth noticing that method (1.1) are not changing if we use the new instead of the old conditions [2]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [25]

$$\xi = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } \ n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1.

(c) As in [2] we can consider methods with memory, if we approximate $b_0 = b_1 = b_2 := T^{(n)} = -[u_{n-1}, x_{n-1}; F]^{-1}, n \geq 1$. This way the order of convergence $4 + \sqrt{19} \approx 8.3589$ was shown in [2, Theorem 2]. The method with memory is defined by $B_0^{(k)} = B_1^{(k)} = B_2^{(k)} = -[u_{n-1}, x_{n-1}; F]^{-1} = -A_{n-1}^{-1}$

$$\begin{aligned} y_n &= x_n - q_1 \\ z_n &= y_n - 3q_2 + 3q_3 - q_4 \\ x_{n+1} &= z_n - 4q_5 + 6q_6 - 4q_7 + q_8, \end{aligned} \tag{2.31}$$

where

$$u_n = x_n + B_0^{(n)}F(x_n),\ h_n = y_n + B_1^{(n)}F(y_n),\ l_n = z_n + B_2^{(n)}F(z_n).$$

In order for us to apply Theorem 2.1 to method (2.31), let us simply replace b_0, b_1, b_2 in Theorem 2.1 by (see (2.7))

$$\frac{\beta_0}{1 - p(\|x_0 - x^*\|)} \leq \frac{\beta_0}{1 - p(r)} := b(r) \tag{2.32}$$

where $\|F'(x^*)^{-1}\| \leq \beta_0$. Indeed, for example by (2.7)

$$|b_2| \leq \| - [u_{n-1}, x_{n-1}; F]^{-1}F'(x^*)F'(x^*)^{-1}\|$$

$$\|[u_{n-1}, x_{n-1}; F]^{-1}F'(x^*)\|\|F'(x^*)^{-1}\| \leq \frac{\beta_0}{1 - p(\|x_0 - x^*\|)} \leq \frac{\beta_0}{1 - p(r)} = b(r). \tag{2.33}$$

The other methods studied in [2] can also be handled in an analogous way as method (2.31).

3. Numerical Examples

The numerical examples are presented in this section. We choose

$$[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta.$$

We have taken $b_0 = 0.5, a_1 = 4, a_2 = -6; a_3 = 4, a_4 = -1$ and $a_5 = 0$.

EXAMPLE 3.1. Let $X = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.9) conditions, we get $w_0(s,t) = \frac{L_0}{2}(s+t), v(t) = \frac{1}{2}(1+e), w(s,t) = \frac{L_0}{2}t, v(s,t) = \frac{1}{2}e^{\frac{1}{L_0}}(s+t)$, $L_0 = e-1$. The parameters are

$$r_1 = 0.3394,\ r_2 = 0.3576,\ r_3 = 0.3086 = r.$$

EXAMPLE 3.2. Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (a) for function v) $w_0(t,s) = w(t,s) = \frac{t+s}{16}$ and $v_0(t) = v(s,t) = 1 + w_0(0,t)$ and $\beta = 1$. Then, the radius of convergence r is given by

$$r_1 = 3.081318457,\ r_2 = 0.99469429615,\ r_3 = 0.1667265158 = r.$$

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