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Dual-Ideal Topological Spaces

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Abstract An ideal in a topological space is a collection of subsets of the space which is closed under finite union and containing every subset of each set that it contains. An ideal topological space is a topological space with an ideal. The concept of ideal topological spaces is studied by Vaidyanathaswamy, Kuratowski, Noiri and many others. In this paper, we define a topology, called dual-ideal topology, on a given ideal topological space and study its properties.

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1. Introduction

Ideals in topological spaces are introduced by Vaidyanathaswamy[13]. Further Noiri[4] and many others[3, 5, 12] developed the concept of ideals in topological spaces. Several ideals and the ideal topologies on the same topological space $(X, \mathcal{T})$ are considered in [12]. Various types of open sets are defined and many topological properties were studied. In 1990, D. Janković and T.R. Hamlett[5] introduced the notion of $\mathcal{I}$-open sets in ideal topological spaces and studied their properties. Julian Dontchev [3] introduced and discussed the concept of pre-$\mathcal{I}$-open sets in 1999. Hatir and Noiri introduced[4] the notion of α-$\mathcal{I}$-open, semi-$\mathcal{I}$-open and β-$\mathcal{I}$-open in ideal topological spaces and further discussed their properties. Some collection of such open sets, such as *semi-$\mathcal{I}$-open sets*, do not form a topology on X.

A collection of subsets of a topological space $(X, \mathcal{T})$ with certain properties is defined as an ideal and a topology $\mathcal{T}_{\mathcal{I}}$ is developed on X so that the members of $\mathcal{I}$ become closed sets in the topology $\mathcal{T}_{\mathcal{I}}$. A collection $\mathcal{G}$ of subsets of a topological space $(X, \mathcal{T})$ with certain properties is defined as a grill[2] and a topology $\mathcal{T}_{\mathcal{G}}$ is developed on X in [11]. From the construction, the topology $\mathcal{T}_{\mathcal{G}}$ induced by a grill $\mathcal{G}$ may seem to have some dual

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properties of the ideal topology $\mathcal{T}_{\mathcal{I}}$. A. Kandil et al.[6] proved that the two topologies $\mathcal{T}_{\mathcal{I}}$ and $\mathcal{T}_{\mathcal{G}}$ are same if $\mathcal{I}$ is defined as the collection of all subsets which are not in $\mathcal{G}$. But in this paper we define a new topology $\mathcal{T}_{\mathcal{D}}$ on X called the dual-ideal topology and study its properties.

In Section 2 we recall some definitions and results from the literature; in Section 3 we define a topology $\mathcal{T}_{\mathcal{D}}$ called dual-ideal topology and discuss some properties and results. In Section 4, we discuss the subspace topologies, product topologies, dual-ideal topologies and their compositions.

2. Preliminaries

Let us start with the definition of an ideal in a topological space.

Definition 2.1. *[13] Let X be any set. A nonempty collection $\mathcal{I}$ of subsets of X satisfying the following*

i. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
ii. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

is called an ideal $\mathcal{I}$ on X. If $\mathcal{I}$ is an ideal on a topological space $(X, \mathcal{T})$, then the triplet $(X, \mathcal{T}, \mathcal{I})$ is called an ideal topological space or ideal space.

The collection $\mathscr{P}(X)$ of all subsets of X and the collection $\{\emptyset\}$ are some trivial examples of ideals. We call $\{\emptyset\}$ as the empty ideal. We note that the empty ideal is not an empty set; it is the collection containing only one element, namely the empty set.

A closure operator on a set X is a function on $\mathscr{P}(X)$ taking A to $\overline{A}$ satisfying the following conditions:

i. $\overline{\emptyset} = \emptyset$
ii. For each A, $A \subseteq \overline{A}$
iii. $\overline{\overline{A}} = \overline{A}$
iv. For any A and B, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

These four conditions are called Kuratowski closure axioms [7]. If "—" is a closure operator on a set X, $\mathcal{F}$ is the family of all subsets A of X for which $\overline{A} = A$, and if $\mathcal{T}$ is the family of complements of members of $\mathcal{F}$, then $\mathcal{T}$ is a topology on X and $\overline{A}$ is the $\mathcal{T}$-closure of A for each subset A of X. This topology is called the topology generated by the closure operator " — ".

Throughout this paper X will denote a topological space, $\mathcal{T}$ will denote a topology on X and $\mathcal{I}$ denote an ideal on X, unless otherwise specified. Members of $\mathcal{T}$ are called $\mathcal{T}$-open sets. If $(X, \mathcal{T})$ is a topological space and $x \in X$, $\mathcal{T}(x)$ denote the set $\{U \in \mathcal{T} / x \in U\}$, the collection of all open sets containing x.

Definition 2.2. *[8] Let A be a subset of $(X, \mathcal{T})$. Define*

$$A^*{}_{(\mathcal{I},\mathcal{T})} = \{x \in X \ / \ U \cap A \notin \mathcal{I} \text{ for every } U \in \mathcal{T}(x)\}.$$

Let $\overline{A} = A \cup A^*{}_{(\mathcal{I},\mathcal{T})}$. Then " — " is a Kuratowski closure operator which gives a topology on X called the topology generated by $\mathcal{I}$ and denoted by $\mathcal{T}_{\mathcal{I}}$. Members of this topology is called an $\mathcal{T}_{\mathcal{I}}$-open sets.

This topology $\mathcal{T}_{\mathcal{I}}$ is usually finer than $\mathcal{T}$. Now let us see the definition of a grill in a topological space.

Definition 2.3. *[11] Let X be any set. A nonempty collection $\mathcal{G}$ of subsets of X satisfying the following*

i. $\emptyset \notin \mathcal{G}$.

ii. *If $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$.*

iii. *If $A \cup B \in \mathcal{G}$, then either $A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

is called a grill on X.

Definition 2.4. *[11] Let $(X, \mathcal{T})$ be a topological space and $\mathcal{G}$ be a grill on X. Let $A \subseteq X$. Define*

$$\Phi_{\mathcal{G}}(A) = \{x \in X \ / \ U \cap A \in \mathcal{G} \text{ for every } U \in \mathcal{T}(x)\}.$$

Let $\psi(A) = A \cup \Phi(A)$. Then "ψ" is a Kuratowski closure operator which gives a topology on X called the topology induced by a grill $\mathcal{G}$ and denoted by $\mathcal{T}_{\mathcal{G}}$. Members of this topology is called an $\mathcal{T}_{\mathcal{G}}$-open sets.

A lot of theory in ideal topological spaces is developed using grills (See for example [1, 10]). But A. Kandil et al. proved that the two concepts coincide (See Theorems 2.1, 2.2, Corollary 2.3 and Remark 2.10 of [6]).

3. Dual-Ideal Topological Spaces

It is easy to prove that the collection $\{U - I \,/\, U \in \mathcal{T}$ and $I \in \mathcal{I}\}$ is a basis for the topology $\mathcal{T}_{\mathcal{I}}$. The basis elements are formed by removing members of $\mathcal{I}$ from members of $\mathcal{T}$. In this paper we are going to define a basis by adjoining members of $\mathcal{I}$ with members of $\mathcal{T}$ and study the topology $\mathcal{T}_{\mathcal{D}}$ generated by this basis. Further the members of $\mathcal{I}$ are closed in $\mathcal{T}_{\mathcal{I}}$; but in the topology $\mathcal{T}_{\mathcal{D}}$ members of $\mathcal{I}$ will be open. These are some of the reasons to call the topology $\mathcal{T}_{\mathcal{D}}$ as dual-ideal topology.

Definition 3.1. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. The topology $\mathcal{T}_{\mathcal{D}}$ generated by the basis $\mathcal{T} \cup \mathcal{I}$ on X is called the dual-ideal topology with respect to the ideal $\mathcal{I}$. Members of this topology are called $\mathcal{D}$-open sets with respect to the ideal $\mathcal{I}$ and their complements are called $\mathcal{D}$-closed sets with respect to the ideal $\mathcal{I}$.*

Remark 3.2. *The collection $\mathscr{B} = \{V \cup I \,/\, V \in \mathcal{T}, I \in \mathcal{I}\}$ of subsets of X is also a basis for the same topology $\mathcal{T}_{\mathcal{D}}$. When there is no ambiguity we simply write $\mathcal{D}$-open instead of writing $\mathcal{D}$-open with respect to the ideal $\mathcal{I}$.*

The following properties can be proved easily.

- Members of $\mathcal{I}$ and members of $\mathcal{T}$ are $\mathcal{D}$-open.
- If $\mathcal{I}$ is the empty ideal, then $\mathcal{T} = \mathcal{T}_{\mathcal{D}}$.
- The topology $\mathcal{T}_{\mathcal{D}}$ is strictly finer than $\mathcal{T}$ whenever $\mathcal{I}$ contains a non open set.
- The concept of $\mathcal{D}$-openness and $\mathcal{T}_{\mathcal{I}}$-openness are independent to each other as seen below.

  Let $X = \{1, 2, 3\}$, $\mathcal{T} = \{\emptyset, X, \{1\}, \{1, 2\}\}$, $\mathcal{I} = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$. Then $\mathcal{T}_{\mathcal{I}} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$ and the $\mathcal{D}$-open sets with respect to $\mathcal{I}$ are $\{\emptyset, X, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}\}$. It is clear that $\{2\}$ is $\mathcal{T}_{\mathcal{I}}$-open but not $\mathcal{D}$-open and also $\{3\}$ is $\mathcal{D}$-open but not $\mathcal{T}_{\mathcal{I}}$-open.
- If $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals on $(X, \mathcal{T})$ such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $\mathcal{T}_{\mathcal{D}_1} \subseteq \mathcal{T}_{\mathcal{D}_2}$.

Theorem 3.3. *Let $\{\mathcal{I}_1\}$ and $\{\mathcal{I}_2\}$ be ideals on $(X, \mathcal{T})$. Let $\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2$. Then*

$$\mathcal{T}_D = \mathcal{T}_{\mathcal{D}_1} \cap \mathcal{T}_{\mathcal{D}_2}.$$

Proof. Considering the bases $\mathscr{B} = \{V \cup I \,/\, V \in \mathcal{T}, I \in \mathcal{I}\}$, $\mathscr{B}_1 = \{V \cup I \,/\, V \in \mathcal{T}, I \in \mathcal{I}_1\}$ and $\mathscr{B}_2 = \{V \cup I \,/\, V \in \mathcal{T}, I \in \mathcal{I}_2\}$ for $\mathcal{T}_\mathcal{D}$, $\mathcal{T}_{\mathcal{D}_1}$ and $\mathcal{T}_{\mathcal{D}_2}$, as $\mathcal{I} \subseteq \mathcal{I}_1$ and $\mathcal{I} \subseteq \mathcal{I}_2$, we see that $\mathcal{T}_D \subseteq \mathcal{T}_{\mathcal{D}_1}$ and $\mathcal{T}_D \subseteq \mathcal{T}_{\mathcal{D}_2}$. Therefore $\mathcal{T}_D \subseteq \mathcal{T}_{\mathcal{D}_1} \cap \mathcal{T}_{\mathcal{D}_2}$.

On the other hand, let $A \in \mathcal{T}_{\mathcal{D}_1} \cap \mathcal{T}_{\mathcal{D}_2}$ and let $x \in A$. Since $\mathscr{B}_1$ and $\mathscr{B}_2$ are the bases of $\mathcal{T}_{\mathcal{D}_1}$ and $\mathcal{T}_{\mathcal{D}_2}$ respectively, there exist $V_1, V_2 \in \mathcal{T}$, $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $x \in V_1 \cup I_1 \subseteq A$ and $x \in V_2 \cup I_2 \subseteq A$. Let $V = V_1 \cup V_2$ and $I = I_1 \cap I_2$. Thus there exist $V \in \mathcal{T}$ and $I \in \mathcal{I}$ such that $x \in V \cup I \subseteq A$. Therefore $A \in \mathcal{T}_\mathcal{D}$ and hence $\mathcal{T}_\mathcal{D} = \mathcal{T}_{\mathcal{D}_1} \cap \mathcal{T}_{\mathcal{D}_2}$. ∎

Let $\{\mathcal{I}_\alpha\}$ be a collection of ideals on a topological space $(X, \mathcal{T})$. In [12], it is proved that the intersection of topologies generated by ideals is larger than the topology generated by the intersection of ideals and an example is given to prove the occurrence of strict inequality. This is not so in the case of dual-ideal topologies. In fact, in the dual-ideal topological theory, the intersection of topologies generated by ideals is equal to the topology generated by the intersection of ideals as seen in the following theorem.

Theorem 3.4. *Let $\{\mathcal{I}_\alpha\}$ be a collection of ideals on $(X, \mathcal{T})$. Let $\mathcal{I} = \cap \mathcal{I}_\alpha$. Then the dual-ideal topology $\mathcal{T}_\mathcal{D}$ with respect to $\mathcal{I}$ on X and the intersection of dual-ideal topologies $\mathcal{T}_{\mathcal{D}_\alpha}$ with respect to $\mathcal{I}_\alpha$ on X are the same. That is,*

$$\mathcal{T}_\mathcal{D} = \cap \mathcal{T}_{\mathcal{D}_\alpha}.$$

Proof. Let $\mathscr{B}$ and $\mathscr{B}_\alpha$ be bases for $\mathcal{T}_\mathcal{D}$ and $\mathcal{T}_{\mathcal{D}_\alpha}$ as described in Remark 3.2. Let $A \in \mathcal{T}_D$. As $\mathcal{I} \subseteq \mathcal{I}_\alpha$, we have $\mathcal{T}_\mathcal{D} \subseteq \mathcal{T}_{\mathcal{D}_\alpha}$ for every α and hence $A \in \cap \mathcal{T}_{\mathcal{D}_\alpha}$.

To prove the another inequality, let $A \in \cap \mathcal{T}_{\mathcal{D}_\alpha}$ and let $x \in A$. Then $A \in \mathcal{T}_{\mathcal{D}_\alpha}$ for every α. Since $\mathscr{B}_\alpha$ is the basis of $\mathcal{T}_{\mathcal{D}_\alpha}$, there exist $U_\alpha \in \mathcal{T}$ and $I_\alpha \in \mathcal{I}$ such that $x \in U_\alpha \cup I_\alpha \subseteq A$. Let $U = \cup U_\alpha$ and $I = \cap I_\alpha$. Thus there exist $U \in \mathcal{T}$ and $I \in \mathcal{I}$ such that $x \in U \cup I \subseteq A$ and hence $A \in \mathcal{T}_\mathcal{D}$. Therefore $\mathcal{T}_\mathcal{D} = \cap \mathcal{T}_{\mathcal{D}_\alpha}$. ∎

The union of a chain of ideals is always an ideal, but the union of an arbitrary collection of ideals need not be an ideal. If $\{\mathcal{I}_\alpha\}$ is a collection of ideals on X, then the collection $\mathscr{P}(X)$ of all subsets of X is an ideal containing all members of all the ideals $\mathcal{I}_\alpha$; so, the family $\mathscr{C}$ of all ideals on X containing all members of all $\mathcal{I}_\alpha$'s is nonempty and hence the intersection of all ideals in $\mathscr{C}$ is the smallest ideal containing all members of all these ideals $\mathcal{I}_\alpha$. So, for any collection of ideals there exists a smallest ideal containing all these ideals.

In [12] it is proved that, if $\{\mathcal{I}_\alpha\}$ is a collection of ideals and if $\mathcal{I}$ is the smallest ideal containing all $\mathcal{I}_\alpha$'s, then $\mathcal{I}$ is the collection of all sets formed by taking the union of finitely many members from $\cup \mathcal{I}_\alpha$.

Theorem 3.5. *Let $\{\mathcal{I}_\alpha\}$ be a collection of ideals on $(X, \mathcal{T})$. Let $\mathcal{I}$ be the smallest ideal containing all $\mathcal{I}_\alpha$'s. Let $\mathcal{T}_{\mathcal{D}_\alpha}$ be the dual-ideal topology with respect to $\mathcal{I}_\alpha$ and $\mathcal{T}_\mathcal{D}$ be the dual-ideal topology with respect to $\mathcal{I}$. Let $\mathcal{T}_0$ be the smallest topology containing all $\mathcal{T}_{\mathcal{D}_\alpha}$'s. Then $\mathcal{T}_\mathcal{D} = \mathcal{T}_0$.*

Proof. As $\{\mathcal{I}_\alpha\}$ is a collection of ideals and $\mathcal{I}$ is the smallest ideal containing $\mathcal{I}_\alpha$'s, $\mathcal{I}_\alpha \subseteq \mathcal{I}$ and hence $\mathcal{T}_{\mathcal{D}_\alpha} \subseteq \mathcal{T}_\mathcal{D}$ for all α. As $\mathcal{T}_0$ is the smallest topology containing all $\mathcal{T}_{\mathcal{D}_\alpha}$'s, $\mathcal{T}_0 \subseteq \mathcal{T}_\mathcal{D}$.

On the other hand, let us assume that $A \in \mathcal{T}_\mathcal{D}$ and $x \in A$. As $\mathcal{T} \cup \mathcal{I}$ is a basis for the topology $\mathcal{T}_\mathcal{D}$, there exists $B \in \mathcal{T} \cup \mathcal{I}$ such that $x \in B \subseteq A$.

If $B \in \mathcal{T}$, then $B \in \mathcal{T}_{\mathcal{D}_\alpha}$ for all α. Thus we see that there is an open set B in $\mathcal{T}_{\mathcal{D}_\alpha}$ such that $x \in B \subseteq A$ and hence $A \in \mathcal{T}_{\mathcal{D}_\alpha}$ for all α. Therefore $A \in \mathcal{T}_0$.

If $B \notin \mathcal{T}$, then $B \in \mathcal{I}$ and hence $B = K_1 \cup K_2 \cup \cdots \cup K_n$ where $K_i \in \mathcal{I}_{\alpha_i}$. As $K_i \in \mathcal{I}_{\alpha_i}$, we have K_i is open in $\mathcal{T}_{\mathcal{D}_{\alpha_i}}$ and hence K_i is open in $\mathcal{T}_0$. As K_i is open in $\mathcal{T}_0$, $B \in \mathcal{T}_0$. Thus we see that there exists an open set B in $\mathcal{T}_0$ such that $x \in B \subseteq A$. Therefore $A \in \mathcal{T}_0$. In both cases, we proved that $\mathcal{T}_D = \mathcal{T}_0$.

∎

It is well known that the union of a collection of topologies need not be a topology. Even if the collection of topologies is a chain, the union need not be a topology. However if $\{\mathcal{T}_{\mathcal{D}_\alpha}\}$ is a chain of topologies, which are dual-ideal topologies generated by a chain of ideals, then the union is a topology as seen in the following corollary.

Corollary 3.6. *Let $\{\mathcal{I}_\alpha\}$ be a chain of ideals. Let $\mathcal{T}_0 = \cup \mathcal{T}_{\mathcal{D}_\alpha}$ and $\mathcal{I} = \cup \mathcal{I}_\alpha$. Then $\mathcal{T}_\mathcal{D} = \mathcal{T}_0$ and hence $\mathcal{T}_0$ is a topology.*

4. Subspaces and Product Spaces

If $\mathcal{I}$ is an ideal on a set X and if $Y \subseteq X$, then $\mathcal{I}_Y = \{A \cap Y \ / \ A \in \mathcal{I}\}$ is an ideal on Y. Let $(X, \mathcal{T})$ be a topological space. Let $Y \subseteq X$ and $\mathcal{I}$ be an ideal on X. Then $(Y, \mathcal{T}_Y)$ is a subspace. From the ideal $\mathcal{I}_Y$ and the topology $\mathcal{T}_Y$, we construct the dual-ideal topology $(\mathcal{T}_Y)_{\mathcal{D}_Y}$ on Y. On the other hand, using the ideal $\mathcal{I}$ and $\mathcal{T}$, we form the dual-ideal topology $\mathcal{T}_\mathcal{D}$ on X; as Y is a subset of X, we have the subspace topology $(\mathcal{T}_\mathcal{D})_Y$ on Y. In the following theorem we prove that these two topologies $(\mathcal{T}_\mathcal{D})_Y$ and $(\mathcal{T}_Y)_{\mathcal{D}_Y}$ on Y are same.

Theorem 4.1. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space, $Y \subseteq X$. Then the subspace topology $(\mathcal{T}_\mathcal{D})_Y$ of the dual-ideal topology $\mathcal{T}_\mathcal{D}$ on Y and the dual-ideal topology $(\mathcal{T}_Y)_{\mathcal{D}_Y}$ of the subspace topology on Y are the same. That is, $(\mathcal{T}_\mathcal{D})_Y = (\mathcal{T}_Y)_{\mathcal{D}_Y}$.*

Proof. Let $\mathcal{T} \cup \mathcal{I}$ and $\mathcal{T}_Y \cup \mathcal{I}_Y$ be bases for $\mathcal{T}_\mathcal{D}$ and $(\mathcal{T}_Y)_{\mathcal{D}_Y}$ as described in Definition 3.1. Let $A \in (\mathcal{T}_\mathcal{D})_Y$ and let $x \in A$. Then $A = U \cap Y$ for some $U \in \mathcal{T}_\mathcal{D}$ in X. As $x \in A$, we have $x \in U \cap Y$. Since $x \in U$ and $U \in \mathcal{T}_\mathcal{D}$, there exist $B \in \mathcal{T} \cup \mathcal{I}$ such that $x \in B \subseteq U$.

If $B \in \mathcal{T}$, then let $V = B \cap Y$. Thus there exists $V \in \mathcal{T}_Y \cup \mathcal{I}_Y$ such that $x \in V \subseteq A$. Therefore $A \in (\mathcal{T}_Y)_{\mathcal{D}_Y}$.

If $B \notin \mathcal{T}$, then $B \in \mathcal{I}$. Let $J = B \cap Y$. Clearly $J \in \mathcal{I}_Y$. Thus there exists $J \in \mathcal{T}_Y \cup \mathcal{I}_Y$ such that $x \in J \subseteq A$. Therefore $A \in (\mathcal{T}_Y)_{\mathcal{D}_Y}$. In both cases, $A \in (\mathcal{T}_Y)_{\mathcal{D}_Y}$.

To prove the other inequality, let $A \in (\mathcal{T}_Y)_{\mathcal{D}_Y}$ and let $x \in A$. Then there exists $B \in \mathcal{T}_Y \cup \mathcal{I}_Y$ such that $x \in B \subseteq A$.

If $B \in \mathcal{T}_Y$, there exists $V_x \in \mathcal{T}$ such that $B = V_x \cap Y$. Let $W = \bigcup_x V_x$. Clearly $A = W \cap Y$ for an open set W in $\mathcal{T}_\mathcal{D}$ in X. Hence $A \in (\mathcal{T}_\mathcal{D})_Y$.

If $B \notin \mathcal{T}_Y$, then $B \in \mathcal{I}_Y$. Thus there exists $I \in \mathcal{I}$ such that $B = I \cap Y$. Clearly I is open in $\mathcal{T}_\mathcal{D}$. Thus there exists $B \in (\mathcal{T}_\mathcal{D})_Y$ such that $x \in B \subseteq A$ and hence $A \in (\mathcal{T}_\mathcal{D})_Y$. In both cases $A \in (\mathcal{T}_\mathcal{D})_Y$. Therefore $(\mathcal{T}_\mathcal{D})_Y = (\mathcal{T}_Y)_{\mathcal{D}_Y}$. ∎

If $\mathscr{A}$ and $\mathscr{B}$ are collections of subsets of sets X and Y, then the collection $\mathscr{A} \times \mathscr{B} = \{A \times B \ / \ A \in \mathscr{A}, B \in \mathscr{B}\}$ is called the product of $\mathscr{A}$ and $\mathscr{B}$. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals on X_1 and X_2, then $\mathcal{I}_1 \times \mathcal{I}_2$ need not be an ideal on $X_1 \times X_2$. However we can associate an ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ on $X \times Y$ in a natural way. The ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ is defined as the smallest ideal containing $\mathcal{I}_1 \times \mathcal{I}_2$ which can be obtained as the intersection of all ideals containing

$\mathcal{I}_1 \times \mathcal{I}_2$. If $\{\mathscr{C}_\alpha\}$ is a collection of subsets of X, then the collection of all ideals containing $\{\mathscr{C}_\alpha\}$ is nonempty as $\mathscr{P}(X)$ is one such ideal, and the intersection of all such ideals is the smallest ideal containing $\{\mathscr{C}_\alpha\}$. We can construct the ideal using the following lemma.

Lemma 4.2. *Let $\mathscr{C}$ be a collection of subsets of X and let $\mathscr{D}$ be the collection of all subsets of all members of $\mathscr{C}$. If $\mathcal{J}$ is the collection of all subsets of X formed by taking the union of finitely many members from $\mathscr{D}$, then $\mathcal{J}$ is an ideal.*

As the proof is routine we skip the proof.

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on X_1 and X_2. Using the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, one can construct the dual-ideal topologies $\mathcal{T}_{1_{\mathcal{D}_1}}$ on X_1 and $\mathcal{T}_{2_{\mathcal{D}_2}}$ on X_2. Let $\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$ be the product topology on $X_1 \times X_2$ obtained in the classical sense. Now $\mathcal{T}_1 \times \mathcal{T}_2$ is the product topology on $X_1 \times X_2$. Using the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$, one can construct the dual-ideal topology $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ on $X_1 \times X_2$. In the following theorem we prove that $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ is contained in $\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$.

Theorem 4.3. *Let $(X_1, \mathcal{T}_1, \mathcal{I}_1)$ and $(X_2, \mathcal{T}_2, \mathcal{I}_2)$ be two ideal topological spaces. Then the dual-ideal topology $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ of the product topology on $X \times Y$ is contained in the product topology $\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$ of the dual-ideal topology on $X \times Y$. That is, $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2} \subseteq \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$.*

Proof. Let $\mathscr{B}_{X \times Y}$ be the basis for $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ as described in Remark 3.2. That is, $\mathscr{B}_{X \times Y}$ consists of the sets of the form $U \cup I$ such that $U \in \mathcal{T}_1 \times \mathcal{T}_2$ and $I \in \mathcal{I}_1 \otimes \mathcal{I}_2$. Let $A \in (\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ and let $(x, y) \in A$. Then there exist $U \in \mathcal{T}_1 \times \mathcal{T}_2$ and $I \in \mathcal{I}_1 \otimes \mathcal{I}_2$ such that $(x, y) \in U \cup I \subseteq A$. Since $I \in \mathcal{I}_1 \otimes \mathcal{I}_2$, we have $I = K_1 \cup K_2 \cup \cdots \cup K_n$ where $K_i \subseteq A_i \times B_i$ and $A_i \in \mathcal{I}_1$, $B_i \in \mathcal{I}_2$.

If $(x, y) \notin I$, then $(x, y) \in U$. Since $U \in \mathcal{T}_1 \times \mathcal{T}_2$ and $(x, y) \in U$, there exist $V_1 \in \mathcal{T}_1$ and $V_2 \in \mathcal{T}_2$ such that $(x, y) \in V_1 \times V_2 \subseteq U$. Then we have $V_1 \in \mathcal{T}_{1_{\mathcal{D}_1}}$ and $V_2 \in \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $(x, y) \in V_1 \times V_2 \subseteq A$. Therefore $A \in \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$.

If $(x, y) \in I$, then $(x, y) \in K_1 \cup K_2 \cup \cdots \cup K_n$ where $K_i \subseteq A_i \times B_i$ and $A_i \in \mathcal{I}_1$, $B_i \in \mathcal{I}_2$. Without loss of generality we assume that $(x, y) \in K_1$. Then $x \in A_1$ and $y \in B_1$. Let us take $W_1 = \{x\}$ and $W_2 = \{y\}$. As W_1 and W_2 are in $\mathcal{I}_1$ and $\mathcal{I}_2$, we have W_1 and W_2 are open in $\mathcal{T}_{1_{\mathcal{D}_1}}$ and $\mathcal{T}_{2_{\mathcal{D}_2}}$. Thus there exist W_1 and W_2 in $\mathcal{T}_{1_{\mathcal{D}_1}}$ and $\mathcal{T}_{2_{\mathcal{D}_2}}$ such that $(x, y) \in W_1 \times W_2 \subseteq A$ and hence $A \in \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$. In both cases, $A \in \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$. Therefore, $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2} \subseteq \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$. ∎

The following example shows that strict inequality may hold in the above theorem.

Example 4.4. *Let $X_1 = \{1, 2\}$, $X_2 = \{1, 3\}$; let $\mathcal{T}_1 = \{\emptyset, X_1, \{2\}\}$, $\mathcal{T}_2 = \{\emptyset, X_2\}$ be topologies on X_1 and X_2. Let $\mathcal{I}_1 = \{\emptyset, \{1\}\}$, $\mathcal{I}_2 = \{\emptyset\}$. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals on X_1 and X_2. Then*

$$\begin{aligned} \mathcal{T}_{1_{\mathcal{D}_1}} &= \{\emptyset, X_1, \{1\}, \{2\}\} \\ \mathcal{T}_{2_{\mathcal{D}_2}} &= \{\emptyset, X_2\} \\ (\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2} &= \{\emptyset, X_1 \times X_2, \{(2,1), (2,3)\}\} \\ \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}} &= \{\emptyset, X_1 \times X_2, \{(1,1), (1,3)\}, \{(2,1), (2,3)\}\}. \end{aligned}$$

Thus $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2} \subsetneq \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$.

Now we have three ways to form new topologies, namely, finding subspace topology, product of topologies and dual-ideal topologies. We denote these operations by the symbols S, P and D. If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$, then by $S(X, \mathcal{T}, Y)$ we mean the subspace topological space $(Y, \mathcal{T}_Y)$ induced by Y on $(X, \mathcal{T})$. If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces, then by $P((X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2))$ we mean the product topological space $(X_1 \times X_2, \mathcal{T}_1 \times \mathcal{T}_2)$. If $(X, \mathcal{T})$ is a topological space and $\mathcal{I}$ be an ideal on X, then by $D(X, \mathcal{T}, \mathcal{I})$ we mean the topological space $(X, \mathcal{T}_{\mathcal{D}})$.

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on them. Then $\mathcal{T}_1 \times \mathcal{T}_2$ is the product topology on $X_1 \times X_2$ and $\mathcal{I}_1 \otimes \mathcal{I}_2$ is an ideal on $X_1 \times X_2$. Using this ideal we form the dual-ideal topology $(\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ on $X_1 \times X_2$; we denote this space as DP ignoring the arguments. On the other hand, we can compute the product topology $(\mathcal{T}_1)_{\mathcal{D}_1} \times (\mathcal{T}_2)_{\mathcal{D}_2}$ on $X_1 \times X_2$; we denote this space as PD. Similarly we can give meanings to $SP, SD, SPD, DPS, \ldots$. With this notation Theorems 4.1 and 4.3 can be stated as $DP \subseteq PD$ and $SD = DS$. The classical theory of topology it is proved that $SP = PS$ (See Theorem 16.3 in [9]).

We have six different permutations, $SPD, SDP, DPS, DSP, PSD, PDS$, of the three operators S, P, D, and thirty different two permutations of these six triplets giving the relationship between them like $SPD \subseteq PDS, PDS \subseteq SPD, SDP \subseteq SPD$ and so on. Now we see all the thirty relations one by one.

In the following theorems we omit the arguments which can be easily understood. For example, we write $SPD = PDS$ instead of writing,

> Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals on $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. Let Y_1 and Y_2 be subsets of X_1 and X_2. Then the topologies $(\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ and $(\mathcal{T}_{1Y_1})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2Y_2})_{\mathcal{D}_{2_{Y_2}}}$ on $Y_1 \times Y_2$ are same.

The theorems which we are going to prove now, with the notations explained above, can be consolidated as

$$PDS = SPD = PSD \supseteq SDP = DPS = DSP$$

Theorem 4.5. *$PDS = SPD$.*

Proof. We have to prove that

$$(\mathcal{T}_{1Y_1})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2Y_2})_{\mathcal{D}_{2_{Y_2}}} = (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}.$$

Let $\mathcal{T}_1 \cup \mathcal{I}_1$, $\mathcal{T}_2 \cup \mathcal{I}_2$, $\mathcal{T}_{1_{Y_1}} \cup \mathcal{I}_{1_{Y_1}}$ and $\mathcal{T}_{2_{Y_2}} \cup \mathcal{I}_{2_{Y_2}}$ be bases for $\mathcal{T}_{1_{\mathcal{D}_1}}$, $\mathcal{T}_{2_{\mathcal{D}_2}}$, $(\mathcal{T}_{1Y_1})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2Y_2})_{\mathcal{D}_{2_{Y_2}}}$ respectively as described in Definition 3.1. Let $W \in (\mathcal{T}_{1Y_1})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2Y_2})_{\mathcal{D}_{2_{Y_2}}}$ and let $(x, y) \in W$. Then there exist open sets U and V are in $(\mathcal{T}_{1Y_1})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2Y_2})_{\mathcal{D}_{2_{Y_2}}}$ such that

$$(x, y) \in U \times V \subseteq W.$$

Since $x \in U$ and $U \in (\mathcal{T}_{1Y_1})_{\mathcal{D}_{1_{Y_1}}}$, there exists $B_1 \in \mathcal{T}_{1Y_1} \cup \mathcal{I}_{1_{Y_1}}$ such that $x \in B_1 \subseteq U$. Similarly, there exists $B_2 \in \mathcal{T}_{2Y_2} \cup \mathcal{I}_{2_{Y_2}}$ such that $y \in B_2 \subseteq V$.

If $B_1 \in \mathcal{T}_{1Y_1}$ and $B_2 \in \mathcal{T}_{2Y_2}$, then there exist open sets $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$ such that $B_1 = G_1 \cap Y_1$ and $B_2 = G_2 \cap Y_2$. Clearly G_1 and G_2 are open in $\mathcal{T}_{1_{\mathcal{D}_1}}$ and $\mathcal{T}_{2_{\mathcal{D}_2}}$. Let $O = (G_1 \times G_2) \cap (Y_1 \times Y_2)$. Thus there exists an open set O in $(\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ such that $(x, y) \in O \subseteq W$ and hence $W \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$.

If $B_1 \in \mathcal{I}_{1_{Y_1}}$ and $B_2 \in \mathcal{I}_{2Y_2}$, there exist $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$ such that $B_1 = I_1 \cap Y_1$ and $B_2 = I_2 \cap Y_2$. Then I_1 and I_2 are open in $\mathcal{T}_{1_{\mathcal{D}_1}}$ and $\mathcal{T}_{2_{\mathcal{D}_2}}$. Let $O = (I_1 \times I_2) \cap (Y_1 \times Y_2)$.

Then there exists an open set O in $(\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ such that $(x, y) \in O \subseteq W$ and hence $W \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$.

If $B_1 \in \mathcal{T}_{1_{Y_1}}$ and $B_2 \in \mathcal{I}_{2_{Y_2}}$, there exist $V_1 \in \mathcal{T}_1$ and $I_2 \in \mathcal{I}_2$ such that $B_1 = V_1 \cap Y_1$ and $B_2 = I_2 \cap Y_2$. Then V_1 and I_2 are open in $\mathcal{T}_{1_{\mathcal{D}_1}}$ and $\mathcal{T}_{2_{\mathcal{D}_2}}$. Let $O = (V_1 \times I_2) \cap (Y_1 \times Y_2)$. Then there exists an open set O in $(\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ such that $(x, y) \in O \subseteq W$ and hence $W \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$.

If $B_1 \in \mathcal{I}_{1_{Y_1}}$ and $B_2 \in \mathcal{T}_{2_{Y_2}}$, then an argument similar to the above case holds. In all cases, we proved that

$$(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}} \subseteq (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}.$$

To prove the other inequality, let us assume that $W \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ and let $(x, y) \in W$. Then there exists $W_1 \in \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $W = W_1 \cap (Y_1 \times Y_2)$. Since $(x, y) \in W$, we have $(x, y) \in W_1$. Thus there exist $U_x \in \mathcal{T}_{1_{\mathcal{D}_1}}$ and $V_y \in \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $(x, y) \in U_x \times V_y \subseteq W_1$. Since $x \in U_x$ and $U_x \in \mathcal{T}_{1_{\mathcal{D}_1}}$, there exists $B_1 \in \mathcal{T}_1 \cup \mathcal{I}_1$ such that $x \in B_1 \subseteq U_x$. Similarly, there exists $B_2 \in \mathcal{T}_2 \cup \mathcal{I}_2$ such that $y \in B_2 \subseteq V_y$.

If $B_1 \in \mathcal{T}_1$ and $B_2 \in \mathcal{T}_2$, then let $G_1 = B_1 \cap Y_1$, $G_2 = B_2 \cap Y_2$. Therefore G_1 and G_2 are open in $\mathcal{T}_{1_{Y_1}}$ and $\mathcal{T}_{2_{Y_2}}$ and hence G_1 and G_2 are open in $(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$. Thus there exist two open sets G_1 and G_2 in $(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$ such that $(x, y) \in G_1 \times G_2 \subseteq W$ and hence $W \in (\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$.

If $B_1 \in \mathcal{I}_1$ and $B_2 \in \mathcal{I}_2$, then let $J_1 = B_1 \cap Y_1$, $J_2 = B_2 \cap Y_2$. Therefore J_1 and J_2 are in $\mathcal{I}_{1_{Y_1}}$ and $\mathcal{I}_{2_{Y_2}}$ and hence J_1 and J_2 are open in $(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$. Thus there exist two open sets J_1 and J_2 in $(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$ such that $(x, y) \in J_1 \times J_2 \subseteq W$ and hence $W \in (\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$.

If $B_1 \in \mathcal{T}_1$ and $B_2 \in \mathcal{I}_2$, then let $G_1 = B_1 \cap Y_1$, $J_2 = B_2 \cap Y_2$. Therefore $G_1 \in \mathcal{T}_{1_{Y_1}}$ and $J_2 \in \mathcal{I}_{2_{Y_2}}$ and hence G_1 and J_2 are open in $(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$. Thus there exist two open sets G_1 and J_2 in $(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}}$ and $(\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$ such that $(x, y) \in G_1 \times J_2 \subseteq W$ and hence $W \in (\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}$.

If $B_1 \in \mathcal{I}_1$ and $B_2 \in \mathcal{T}_2$, then an argument similar to the above case holds. In all cases, we proved that

$$W \in (\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}}.$$

Therefore

$$(\mathcal{T}_{1_{Y_1}})_{\mathcal{D}_{1_{Y_1}}} \times (\mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{2_{Y_2}}} = (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}.$$

∎

Theorem 4.6. *$SPD = PSD$.*

Proof. We have to prove that

$$(\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2} = (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}.$$

Let $W \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ and let $(x, y) \in W$. Then, there exists $W_1 \in \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $W = W_1 \cap (Y_1 \times Y_2)$. Since $(x, y) \in W_1$, there exist $U_x \in \mathcal{T}_{1_{\mathcal{D}_1}}$ and $V_y \in \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $(x, y) \in U_x \times V_y \subseteq W_1$. Let $G_1 = U_x \cap Y_1$ and $G_2 = V_y \cap Y_2$. Thus there exist open sets G_1 and G_2 in $(\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1}$ and $(\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$ such that $(x, y) \in G_1 \times G_2 \subseteq W$ and hence $W \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$.

On the other hand, let us assume that $W \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$ and let $(x, y) \in W$. Then there exist $U_x \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1}$ and $V_y \in (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$ such that $(x, y) \in U_x \times V_y \subseteq W$.

Since $U_x \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1}$ and $V_y \in (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$, there exist $G_x \in \mathcal{T}_{1_{\mathcal{D}_1}}$ and $H_y \in \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $U_x = G_x \cap Y_1$ and $V_y = H_y \cap Y_2$. Let $G = \cup G_x$ and $H = \cup H_y$. Clearly G and H are in $\mathcal{T}_{1_{\mathcal{D}_1}}$ and $\mathcal{T}_{2_{\mathcal{D}_2}}$. Let $O = (G \times H) \cap (Y_1 \times Y_2)$. Since $G \times H$ is open in $\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$, $O \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$. Thus there exists an open set $O \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$ such that $(x, y) \in O \subseteq W$ and hence $W \in (\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2}$. Therefore

$$(\mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}})_{Y_1 \times Y_2} = (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}.$$

∎

Theorem 4.7. *SDP ⊆ PSD.*

Proof. We have to prove that

$$((\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2})_{Y_1 \times Y_2} = (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}.$$

Let $W \in ((\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2})_{Y_1 \times Y_2}$ and let $(x, y) \in W$. Then there exists $W_1 \in (\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ such that $W = W_1 \cap (Y_1 \times Y_2)$. Since $W_1 \in (\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2}$ and by Theorem 4.3, we have $W_1 \in \mathcal{T}_{1_{\mathcal{D}_1}} \times \mathcal{T}_{2_{\mathcal{D}_2}}$. Since $(x, y) \in W_1$, there exist $U_1 \in \mathcal{T}_{1_{\mathcal{D}_1}}$ and $V_1 \in \mathcal{T}_{2_{\mathcal{D}_2}}$ such that $(x, y) \in U_1 \times V_1 \subseteq W_1$. Let $U = U_1 \cap Y_1$ and $V = V_1 \cap Y_2$. Then $U \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1}$ and $V \in (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$. Since $U_1 \times V_1 \subseteq W_1$, we have $(U_1 \cap Y_1) \times (V_1 \cap Y_2) = U \times V \subseteq W$. Thus we get open sets $U \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1}$ and $V \in (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$ such that $(x, y) \in U \times V \subseteq W$ and hence $W \in (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$. Therefore

$$((\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2})_{Y_1 \times Y_2} \subseteq (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}.$$

∎

The following example shows that strict inequality may hold in the above theorem.

Example 4.8. *Let* $X_1 = \{1, 2, 3\}$, $X_2 = \{1, 2, 4\}$, $Y_1 = \{1, 3\}$ *and* $Y_2 = \{1, 2\}$; *let* $\mathcal{T}_1 = \{\emptyset, X_1, \{1\}, \{1, 2\}\}$, $\mathcal{T}_2 = \{\emptyset, X_2, \{1\}\}$ *be topologies on* X_1 *and* X_2. *Let* $\mathcal{I}_1 = \{\emptyset, \{3\}\}$, $\mathcal{I}_2 = \{\emptyset, \{4\}\}$; *then* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are ideals on* X_1 *and* X_2. *Then*

$$((\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2})_{Y_1 \times Y_2} = \left\{ \begin{array}{l} \emptyset, Y_1 \times Y_2, \{(1,1), (1,2), (3,1)\} \\ \qquad \{(1,1), (3,1)\}, \{(1,1), (1,2)\}, \{(1,1)\} \end{array} \right\}$$

$$(\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2} = \left\{ \begin{array}{l} \emptyset, Y_1 \times Y_2, \{(1,1), (1,2), (3,1), (3,2)\}, \\ \quad \{(1,1), (1,2), (3,1)\}, \{(1,1), (3,1), (3,2)\}, \\ \quad \{(1,1), (3,1)\}, \{(1,1), (1,2)\}, \{(3,1), (3,2)\}, \\ \quad \{(1,1)\}, \{(3,1)\}. \end{array} \right\}$$

Thus $((\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2})_{Y_1 \times Y_2} \subsetneq (\mathcal{T}_{1_{\mathcal{D}_1}})_{Y_1} \times (\mathcal{T}_{2_{\mathcal{D}_2}})_{Y_2}$.

Theorem 4.9. *SDP = DSP.*

Proof. From Theorem 4.1, it follows that

$$((\mathcal{T}_1 \times \mathcal{T}_2)_{\mathcal{D}_1 \times \mathcal{D}_2})_{Y_1 \times Y_2} = ((\mathcal{T}_1 \times \mathcal{T}_2)_{Y_1 \times Y_2})_{\mathcal{D}_{1_{Y_1}} \times \mathcal{D}_{2_{Y_2}}}.$$

∎

Theorem 4.10. *DSP = DPS.*

Proof. In the classical theory of topology it is proved that $SP = PS$ (See Theorem 16.3 in [9]). That is

$$(\mathcal{T}_1 \times \mathcal{T}_2)_{Y_1 \times Y_2} = \mathcal{T}_{1_{Y_1}} \times \mathcal{T}_{2_{Y_2}} \tag{4.1}$$

From this it follows that

$$((\mathcal{T}_1 \times \mathcal{T}_2)_{Y_1 \times Y_2})_{\mathcal{D}_{1_{Y_1}} \times \mathcal{D}_{2_{Y_2}}} = (\mathcal{T}_{1_{Y_1}} \times \mathcal{T}_{2_{Y_2}})_{\mathcal{D}_{1_{Y_1}} \times \mathcal{D}_{2_{Y_2}}}.$$

∎

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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