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Eco–epidemiological prey–predator model for susceptible–infected species

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Abstract This paper aims to study the dynamical behavior of a prey–predator system where both prey and predator populations are affected by disease with susceptible–infected. We also analysis the system of the equilibrium point and stability analysis. We derive the boundedness. A system of four differential equation susceptible–infected prey species and predator species has been proposed and analyzed. Computer simulations are carried out to illustrate our analytical findings. In population ecology, in particular, the predator–prey interaction in presence of an eco–epidemiological system of the biological implications of analytical and numerical findings are discussed critically.

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1. Introduction

The study of prey–predator effect of dynamics with an susceptible–infected both prey predator has a great important in ecological the prey–predator population[12]. An infected place is one where germs or bacteria are causing a disease to spread among people or animals. But this area has been neglected for a long time in theoretical ecology. Recently a few researchers have cultured some prey–predator models for disease with susceptible–infected [3]. Ecological populations suffer from various diseases. The effect of disease in ecological system is an important area from mathematical model[5]. So, in recent time ecologists and researchers are paying more and more affection to the development of important tool along with experimental ecology.
The literature abounds with such evidences, in the last few decades, mathematical models have become extremely important tools in understanding and analyzing this spread and susceptible–infected control of infection diseases [7]. They established that ecological population suffer from various diseases [10], [9]. The importance of viruses for marine and

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especially phytoplankton ecology has been acknowledged in several recent publications using predation may defeat spatial spread of infection [4]. A prey–predator model with infection in both prey and predator for numerous models have been cultured by various researchers [6]. The dynamics in a harvested prey–predator model with susceptible–infected –susceptible epidemic diseases in the prey, both the theoretical ecology and the epidemiology are developed research field and treated separately[2], [1]. An epidemic model includes the property of population growth, the spread rules of infectious diseases and the related ecological factors to construct mathematical model reflecting the dynamic properties of infection for effects of additional food in a susceptible and infected prey–predator model has been formulated [8], [11]. Before presenting the another extension involves treating the system with stage structure which is compartmentalizing a species class into mature and immature class. Many researchers study the prey and predator models where there are stage structures in [15], [16]. The Leslie –Grower model with type II functional response is represented by switching from simple to complex dynamics in a predator–prey parasite model. For an interplay between infection rate and incubation delay [13]. Eco–epidemiology is a branch in mathematical biology which considers both the ecological and epidemiological issues simultaneously. It is well known that the prey–predator harvesting as a disease control measure in an eco–epidemiological system[14]. Now we will present some examples of the role of environmental disturbance in an eco–epidemiological model with disease from external source [4]. Viral, bacterial and parasitic for in recent times, harvesting is an important issue in the predator–prey system where both species are infected by some transmissible diseases [6]. A study of harvesting in a predator–prey model with disease in both populations.

2. Method for selection of parameter values

The methodology to select parameter values for simulation experiments is based on the dynamical representation of a given model. For example consider the 4D model system given below, obtained by coupling the RM model with the Leslie–Gower model, which is schematically. Most real–world problems are highly nonlinear and a large number of them can be modeled in the form of a system of nonlinear ordinary differential equations with computer simulations of such mathematical models are being used extensively to solve such problems.

The process of mathematical modeling can be divided into three major steps as follows

(1) Obtain a clear idea of the various types of laws governing the problem.

(2) Idealize or simplify the problem by introducing certain assumptions and to convert the problem into mathematical equations.

(3) Solve the mathematical equations and interpret the results, this requires analysis of analytical, numerical and graphical tools.

We make the following assumptions to formulate the mathematical model assumption:

(i) We have considered ”eco–epidemiology” a prey–predator ecosystem where the total prey–predator population densities are denoted by M and N respectively. It is assumed that both the prey and the predator are susceptible to some transmissible disease like viral disease.

(ii) We have considered an eco–epidemiology prey–predator populations are divided into two classes (1) Susceptible prey (W) and infected prey (x) and (2) Susceptible predator (y) and infected predator (z).

(iii) We have assumed that the prey species is a commutative species and susceptible prey (W) is capable of reproducing with logistic law having carrying capacity K_1 and intrinsic growth rate r_1. Infected prey (x) is capable of reproducing with logistic law having carrying capacity K_2 and intrinsic growth rate r_2.

$$\frac{dW}{dt} = W\left(r_1\left(1-\frac{W}{K_1}\right)\right), \frac{dx}{dt} = x\left(r_2\left(1-\frac{x}{K_2}\right)\right)$$

Assumption of the model of the prey–predator for susceptible–infected species are

$$\begin{aligned}\frac{dW}{dt} &= W\left(r_1\left(1-\frac{W}{K_1}\right) - c_1\, y - a_1 z - \alpha\, x\right)\\ \frac{dx}{dt} &= x\left(r_2\left(1-\frac{x}{K_2}\right) - c_2\, y - a_2 z + \alpha\, W\right)\\ \frac{dy}{dt} &= y\left(c_3 W - \beta\, z + a_3 x - d_1\right)\\ \frac{dz}{dt} &= z\left(c_4 W + \beta\, y + a_4 x - d_2\right)\end{aligned} \tag{2.1}$$

Let r_1, r_2 are the intrinsic growth rate of the susceptible prey and infected prey, respectively. Let c_1, c_2 are the capture rate of the susceptible prey and infected prey by the susceptible predator, respectively. Let a_1, a_2 are the capture rate of the susceptible prey and infected prey by the infected predator, respectively. Let c_3, c_4 are the conversion factors for the susceptible predator and the infected predator, respectively, due to consumption of the susceptible prey. Let a_3, a_4 are the conversion factors for the susceptible predator and the infected predator, respectively, due to consumption of the infected prey. Let d_1, d_2 are the over–crowding in the susceptible predator and infected predator respectively. Let α is force of infection between the susceptible prey and the infected prey. Let β is force of infection between the susceptible predator and the infected predator.

3. Range of the interval in susceptible and infected species

In this model, we consider biological phenomena parameters that are imprecise in nature.

An interval number A is a closed interval $[a_l, a_r]$ and is defined by

$$A = \{\alpha : a_l \le \alpha \le a_r, \alpha \in R\}$$

where R is the set of real numbers and a_l, a_r are the left and right limits of the interval number, respectively.

An interval–valued number $\widehat{a}$ on [0, 1] is a closed subinterval of [0,1] that is $\widehat{a} = [a_l, a_u]$ such that $0 \le a_l \le a_u \le 1$, where a_l and a_u are the lower and upper limits of $\widehat{a}$, respectively. In this notation, $\widehat{0} = [0, 0]$ and $\widehat{1} = [1, 1]$. For any two interval numbers $\widehat{a} = [a_l, a_u]$ and $\widehat{b} = [b_l, b_u]$ on $[0, 1]$ we define

$$\widehat{a} \le \widehat{b} \Leftrightarrow a_l \le b_l \ and \ a_u \le b_u$$

$$\widehat{a} = \widehat{b} \Leftrightarrow a_l = b_l \ and \ a_u = b_u$$

Before the work is interval valued function defined. present this work is very different and not for the function define just use in the range of interval [0, 1] is α, β for force of infection between the susceptible prey and infected prey, force of infection between the susceptible predator and the infected predator respectively.

4. Boundedness results

In this section we have proof of the boundedness theorem.

Theorem 4.1 Both prey are always bounded above for $r_1 > 0, K_1 > 0$.

Proof If W(0)=0, then the result is trivial, if $W(0) > 0$, Then $W(t) > 0$ for all t on adding equation (2.1) we obtain

$$\frac{dW}{dt} \leq r_1 W(1 - \frac{W}{K_1})$$

$$\frac{dx}{dt} \leq r_2 x(1 - \frac{x}{K_2}),\ \limsup_{t\longrightarrow\infty} W(t) \leq K_1,\ \limsup_{t\longrightarrow\infty} x(t) \leq K_2$$

Theorem 4.2 Both predator are always bounded above

Proof If $y(0) = 0$ the result is obvious.

we obtain the equation (2.1) If $y(0) > 0$, then $\frac{dy}{dt} < 0\ if\ d_1 y > 1$ $\frac{dz}{dt} < 0\ if\ d_2 z > 1$ $\limsup_{t\longrightarrow\infty} y(t) \leq \frac{1}{d_1}$, $\limsup_{t\longrightarrow\infty} z(t) \leq \frac{1}{d_2}$

Theorem 4.3 The trajectories of system (2.1) are bounded.

Proof Define the function $l = W + x + y + z$ and take its time derivative along the solution of (2.1)

$$\frac{dl}{dt} = \frac{dW}{dt} + \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}$$

now $\frac{dl}{dt} + \rho l = r_1 W(1 - \frac{W}{k_1}) + r_2 x(1 - \frac{x}{k_2}) - d_1 y - d_2 z + \rho W + \rho x + \rho y + \rho z + (\rho + r_1)W + (\rho + r_2)x + (\rho - d_1)y + (\rho - d_2)z - \frac{r_1 W^2}{k_1} - \frac{r_2 x^2}{k_2}$

where ρ is a positive constant for $\rho > d_1\ or\ \rho > d_2\ \rho(d_1 + d_2)\ given\ \epsilon > 0$ there exists to such that t on $t \geq t_0$

$\frac{dl}{dt} + \rho l \leq m + \epsilon, m = \min\{(\rho + r_1), (\rho + r_2), (\rho - d_1), (\rho - d_2)\}$

Hence $\frac{d}{dt}(le^{\rho t}) \leq (m + \epsilon)e^{\rho t}$ $l(t) \leq l(t_0)e^{-\rho(t-t_0)} + \frac{(m+t)}{\rho}(1 - e^{-\rho(t-t_0)})$

Letting $t \longrightarrow 0$ then letting $\epsilon \longrightarrow 0$

$$\limsup_{t\longrightarrow\infty} l(t) \leq \frac{m}{\rho}$$

On the initial conditions. Hence the system (2.1) are bounded.

5. Analytical solution of critical point and stability analysis

The equilibrium point of the parametric model (2.1) is given by steady state equations $\frac{dW}{dt} = \frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$. The system has 13 equilibrium points, and after algebraic calculation we get the trivial, axial and non–trivial equilibrium points as follows.

(1) The trivial equilibrium point are

$$\Xi_1 \{W = 0, x = 0, y = 0, z = 0\}$$

(2) The infected prey–free and both predator–free equilibrium point are

$$\Xi_2 \{W = K_1, x = 0, y = 0, z = 0\}$$

(3) Susceptible prey–free and both predator–free equilibrium point are

$$\Xi_3 \{W = 0, x = K_2, y = 0, z = 0\}$$

(4) Infected prey predator–free equilibrium point are

$$\Xi_4\left\{W=\frac{d_1}{c_3},x=0,y=\frac{1}{c_1}r_1\left(\frac{K_1c_3-d_1}{K_1c_3}\right),z=0\right\}$$

(5) Infected prey and susceptible predator–free equilibrium point are

$$\Xi_5\left\{W=\frac{d_2}{c_4},x=0,y=0,z=\frac{1}{a_1}r_1\left(\frac{K_1c_4-d_2}{K_1c_4}\right)\right\}$$

(6) Susceptible prey and infected predator–free equilibrium point are

$$\Xi_6\left\{W=0,x=\frac{d_1}{a_3},y=\frac{1}{c_2}r_2\left(\frac{K_2a_3-d_1}{K_2a_3}\right),z=0\right\}$$

(7) susceptible prey–predator free equilibrium point are

$$\Xi_7\left\{W=0,x=\frac{d_2}{a_4},y=0,z=\frac{1}{a_2}r_2\left(\frac{K_2a_4-d_2}{K_2a_4}\right)\right\}$$

(8) Both prey free equilibrium point are

$$\Xi_8\left\{W=0,x=0,y=\frac{d_2}{\beta},z=-\frac{d_1}{\beta}\right\}$$

(9) Infected predator free equilibrium point are

$$\Xi_9\left\{W=\vartheta_1,x=\vartheta_2,y=\vartheta_3,z=0\right\}$$

Where

$$\vartheta_1=-\frac{-a_3c_2k_1k_2r_1+a_3c_1k_1k_2r_2+\alpha c_2d_1k_1k_2-c_1d_1k_1r_2}{\alpha a_3c_1k_1k_2+a_3c_2k_2r_1-\alpha c_2c_3k_1k_2+c_1c_3k_1r_2},$$

$$\vartheta_2=-\frac{-\alpha c_1d_1k_1k_2-c_2d_1k_2r_1+c_2c_3k_1k_2r_1-c_1c_3k_1k_2r_2}{\alpha a_3c_1k_1k_2+a_3c_2k_2r_1-\alpha c_2c_3k_1k_2+c_1c_3k_1r_2},$$

$$\vartheta_3=-\frac{-a_3\alpha k_1k_2r_1-a_3k_2r_1r_2+\alpha c_3k_1k_2r_2-c_3k_1r_1r_2+\alpha^2d_1k_1k_2+d_1r_1r_2}{\alpha a_3c_1k_1k_2+a_3c_2k_2r_1-\alpha c_2c_3k_1k_2+c_1c_3k_1r_2}$$

(10) Infected prey free equilibrium point are

$$\Xi_{10}\left\{W=\vartheta_4,x=0,y=\vartheta_5,z=\vartheta_6\right\}$$

Where

$$\vartheta_4=-\frac{-a_1d_1k_1+c_1d_2k_1-\beta k_1r_1}{a_1c_3k_1-c_1c_4k_1+\beta r_1},$$

$$\vartheta_5=-\frac{a_1c_4d_1k_1-a_1c_3d_2k_1+\beta c_4k_1r_1-\beta d_2r_1}{\beta\left(a_1c_3k_1-c_1c_4k_1+\beta r_1\right)},$$

$$\vartheta_6=-\frac{-c_1c_4d_1k_1+c_1c_3d_2k_1-\beta c_3k_1r_1+\beta d_1r_1}{\beta\left(a_1c_3k_1-c_1c_4k_1+\beta r_1\right)}$$

(11) susceptible predator free equilibrium point are

$$\Xi_{11}\left\{W=\vartheta_7,x=\vartheta_8,y=0,z=\vartheta_9\right\}$$

Where

$$\vartheta_7=-\frac{\alpha a_2d_2k_1k_2-a_1d_2k_1r_2-a_2a_4k_1k_2r_1+a_1a_4k_1k_2r_2}{-\alpha a_2c_4k_1k_2+a_1c_4k_1r_2+\alpha a_1a_4k_1k_2+a_2a_4k_2r_1},$$

$$\vartheta_8=-\frac{a_2c_4k_1k_2r_1-a_1c_4k_1k_2r_2-\alpha a_1d_2k_1k_2-a_2d_2k_2r_1}{-\alpha a_2c_4k_1k_2+a_1c_4k_1r_2+\alpha a_1a_4k_1k_2+a_2a_4k_2r_1}$$

$$\vartheta_9=-\frac{-a_4\alpha k_1k_2r_1-a_4k_2r_1r_2+\alpha c_4k_1k_2r_2-c_4k_1r_1r_2+\alpha^2d_2k_1k_2+d_2r_1r_2}{-\alpha a_2c_4k_1k_2+a_1c_4k_1r_2+\alpha a_1a_4k_1k_2+a_2a_4k_2r_1}$$

(12) Susceptible prey free equilibrium point are

$$\Xi_{12}\left\{W=0,x=\vartheta_{10},y=\vartheta_{11},z=\vartheta_{12}\right\}$$

where

$$\vartheta_{10}=-\frac{-a_2d_1k_2+c_2d_2k_2-\beta k_2r_2}{-a_4c_2k_2+a_2a_3k_2+\beta r_2},$$

$$\vartheta_{11}=-\frac{a_2a_4d_1k_2-a_2a_3d_2k_2+a_4\beta k_2r_2-\beta d_2r_2}{\beta\left(-a_4c_2k_2+a_2a_3k_2+\beta r_2\right)},$$

$$\vartheta_{12} = -\frac{-a_4 c_2 d_1 k_2 + a_3 c_2 d_2 k_2 - a_3 \beta k_2 r_2 + \beta d_1 r_2}{\beta\left(-a_4 c_2 k_2 + a_2 a_3 k_2 + \beta r_2\right)}$$

(13) Non–trivial equilibrium point

$$\Xi_{13}\left\{W = \overline{W}, x = \overline{x}, y = \overline{y}, z = \overline{z}\right\}.$$

The system of the equation (2.1) is Jacobian matrix given by

$$\begin{bmatrix} m_{11} & -\alpha W & -W c_1 & -a_1 W \\ \alpha x & m_{22} & -x c_2 & -a_2 x \\ y c_3 & y a_3 & m_{33} & -\beta y \\ z c_4 & z a_4 & \beta z & m_{44} \end{bmatrix}$$

Where $m_{11} = r_1 - 2\frac{r_1 W}{K_1} - c_1 y - a_1 z - \alpha x, m_{22} = r_2 - 2\frac{r_2 x}{K_2} - c_2 y - a_2 z + \alpha W, m_{33} = c_3 W - \beta z + a_3 x - d_1, m_{44} = c_4 W + \beta y + a_4 x - d_2$

6. Nature of the equilibrium and stability analysis

In this section we shall discuss the stability properties of the critical point.

Theorem 6.1 Given the linearized system of equations (2.1) is trivial equilibrium point.

In which equilibrium point $\Xi_1(0,0,0,0)$ is saddle point.

Proof The variation of the Jacobian matrix are

$$J_0 = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & -d_1 & 0 \\ 0 & 0 & 0 & -d_2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = r_1, \lambda_2 = r_2, \lambda_3 = -d_1, \lambda_4 = -d_2$.

(1) An equilibrium point $\Xi_1(0,0,0,0)$ is called a saddle point. If all eigenvalues of matrix J_0 have nonzero real parts is called a hyperbolic equilibrium point exists. Then the eigenvalues of matrix J_0 has at least of eigenvalues with a positive real parts and at least one eigenvalues with a negative real part is called a saddle point. Therefore the eigenvalues $\lambda_1 = r_1 > 0, \lambda_2 = r_2 > 0, \lambda_3 = -d_1 < 0, \lambda_4 = -d_2 < 0$ and that is $r_1 > 0, r_2 > 0, d_1 < 0, d_1 < 0$ is a saddle point.

Theorem 6.2 Given the linearized system of equations (2.1) is infected prey–free and both predator–free equilibrium point. In which the equilibrium point $\Xi_2(K_1,0,0,0)$ are source and saddle point.

Proof The variation of the Jacobian matrix are

$$J_1 = \begin{bmatrix} -r_1 & -\alpha K_1 & -K_1 c_1 & -a_1 K_1 \\ 0 & r_1 + \alpha K_1 & 0 & 0 \\ 0 & 0 & K_1 c_3 - d_1 & 0 \\ 0 & 0 & 0 & K_1 c_4 - d_2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = r_1 + \alpha K_1, \lambda_2 = K_1 c_4 - d_2, \lambda_3 = K_1 c_3 - d_1, \lambda_4 = -r_1$.

(1) An equilibrium point $\Xi_2(K_1,0,0,0)$ is called a sink. Since all of the eigenvalues of matrix J_1 have negative real parts. Therefore the eigenvalues $\lambda_1 = r_1 + \alpha K_1 > 0, \lambda_2 = K_1 c_4 - d_2 > 0, \lambda_3 = K_1 c_3 - d_1 > 0, \lambda_4 = -r_1 > 0$ and that is $K_1 c_4 > d_2, K_1 c_3 > d_1, r_1 < 0$ is a sink.

(2) An equilibrium point $\Xi_2(K_1, 0, 0, 0)$ is called a saddle point. If all eigenvalues of matrix J_1 have nonzero real parts is called a hyperbolic equilibrium point exists. Then the eigenvalues of matrix J_1 has at least of eigenvalues with a positive real parts and at least one eigenvalues with a negative real part is called a saddle point. Therefore the eigenvalues $\lambda_1 = r_1 + \alpha K_1 > 0, \lambda_2 = K_1 c_4 - d_2 > 0, \lambda_3 = K_1 c_3 - d_1 > 0, \lambda_4 = -r_1 < 0$ and that is $K_1 c_4 > d_2, K_1 c_3 > d_1, r_1 > 0$ is a saddle point.

Theorem 6.3 Given the linearized system of equations (2.1) is susceptible prey–free and both predator–free equilibrium point. Then the equilibrium point $\Xi_3(0, K_2, 0, 0)$ are source and saddle point.

Proof The variation of the Jacobian matrix are

$$J_2 = \begin{bmatrix} -\alpha K_2 + r_1 & 0 & 0 & 0 \\ \alpha K_2 & -r_2 & K_2 c_2 & -K_2 a_2 \\ 0 & 0 & K_2 a_3 - d_1 & 0 \\ 0 & 0 & 0 & K_2 a_4 - d_2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -r_2, \lambda_2 = K_2 a_4 - d_2, \lambda_3 = K_2 a_3 - d_1, \lambda_4 = -\alpha K_2 + r_1$.

(1) An equilibrium point $\Xi_3(0, K_2, 0, 0)$ is called a sink. Since all of the eigenvalues of matrix J_2 have negative real parts. Therefore the eigenvalues $\lambda_1 = -r_2 > 0, \lambda_2 = K_2 a_4 - d_2 > 0, \lambda_3 = K_2 a_3 - d_1 > 0, \lambda_4 = -\alpha K_2 + r_1 > 0$ and that is $r_2 < 0, K_2 a_4 > d_2, K_2 a_3 > d_1, r_1 > \alpha K_2$ is source.

(2) An equilibrium point $\Xi_3(0, K_2, 0, 0)$ is called a saddle point. If all eigenvalues of matrix J_2 have nonzero real parts is called a hyperbolic equilibrium point exists. Then the eigenvalues of matrix J_2 has at least of eigenvalues with a positive real parts and at least one eigenvalues with a negative real part is called a saddle point. Therefore the eigenvalues $\lambda_1 = -r_2 > 0, \lambda_2 = K_2 a_4 - d_2 > 0, \lambda_3 = K_2 a_3 - d_1 > 0, \lambda_4 = -\alpha K_2 + r_1 < 0$ and that is $r_2 < 0, K_2 a_4 > d_2, K_2 a_3 > d_1, r_1 > \alpha K_2$ is a saddle point.

Theorem 6.4 Given the linearized system of equations (2.1) is infected prey predator–free equilibrium point. Then the equilibrium point $(W = \frac{d_1}{c_3}, x = 0, y = \frac{1}{c_1} r_1, z = 0)$ where $\tau_1 = \left(\frac{K_1 c_3 - d_1}{K_1 c_3}\right)$ is locally asymptotically stable if the following conditions hold as follows: $c_2 r_2 \tau_1 c_3 > c_1(\alpha d_1 + c_3 r_2), \beta r_1 \tau_1 c_3 > c_1(c_3 d_2 - c_4 d_1)$.

Proof

$$J_4 = \begin{bmatrix} r_1 - 2\,\frac{d_1 r_1}{K_1 c_3} - \tau_1 r_1 & -\frac{\alpha\, d_1}{c_3} & -\frac{d_1 c_1}{c_3} & -\frac{a_1 d_1}{c_3} \\ 0 & r_2 - \frac{c_2\, r_1\, \tau_1}{c_1} + \frac{\alpha\, d_1}{c_3} & 0 & 0 \\ \frac{\tau_1 r_1\, c_3}{c_1} & \frac{\tau_1 r_1\, a_3}{c_1} & 0 & -\frac{\beta\, r_1\, \tau_1}{c_1} \\ 0 & 0 & 0 & \frac{c_4 d_1}{c_3} + \frac{\beta\, r_1\, \tau_1}{c_1} - d_2 \end{bmatrix}$$

The corresponding eigenvalues are

$$\lambda_1 = -\frac{c_2\, r_1\, \tau_1 c_3 - \alpha\, c_1\, d_1 - r_2\, c_1\, c_3}{c_1\, c_3},$$

$$\lambda_2 = \frac{\beta\, r_1\, \tau_1 c_3 - c_1\, c_3 d_2 + c_1\, c_4 d_1}{c_1\, c_3},$$

$\lambda_3 = 1/2\,\dfrac{-r_1\,(\tau_1 K_1 c_3 - K_1 c_3 + 2\,d_1) + \sqrt{r_1\,A}}{K_1 c_3}$

$\lambda_4 = -1/2\,\dfrac{r_1\,(\tau_1 K_1 c_3 - K_1 c_3 + 2\,d_1) + \sqrt{r_1\,A}}{K_1 c_3}$ Where $A = -2\,\tau_1 r_1\,{K_1}^2{c_3}^2 - 4\,\tau_1 {K_1}^2{c_3}^2 d_1 + {\tau_1}^2{K_1}^2{c_3}^2 + r_1\,{K_1}^2{c_3}^2 + 4\,\tau_1 r_1\,K_1 c_3 d_1 - 4\,r_1\,K_1 c_3 d_1 + 4\,r_1\,{d_1}^2$ Hence the equilibrium point (4) is locally asymptotically stable if $c_2 r_2 \tau_1 c_3 > c_1(\alpha d_1 + c_3 r_2), \beta r_1 \tau_1 c_3 > c_1(c_3 d_2 - c_4 d_1)$.

Theorem 6.5 Given the linearized system of equations (2.1) is infected prey free and susceptible predator free. In which the equilibrium point $\left\{W = \frac{d_2}{c_4}, x = 0, y = 0, z = \frac{1}{a_1} r_1 \tau_2\right\}$ where $\tau_2 = \left(\frac{K_1 c_4 - d_2}{K_1 c_4}\right)$ is locally asymptotically stable if $a_2 r_2 \tau_2 c_4 > a_1(\alpha d_2 + r_2 c_4), \beta r_1 \tau_2 c_4 > a_1(d_2 a_3 - d_1 c_4)$.

Proof

$$J_5 = \begin{bmatrix} r_1 - 2\,\frac{d_2 r_1}{K_1 c_4} - r_1\,\tau_2 & -\frac{\alpha\,d_2}{c_4} & -\frac{d_2 c_1}{c_4} & -\frac{a_1 d_2}{c_4} \\ 0 & r_2 - \frac{a_2 r_1\,\tau_2}{a_1} + \frac{\alpha\,d_2}{c_4} & 0 & 0 \\ 0 & 0 & \frac{c_3 d_2}{c_4} - \frac{\beta\,r_1\,\tau_2}{a_1} - d_1 & 0 \\ \frac{r_1\,\tau_2 c_4}{a_1} & \frac{r_1\,\tau_2 a_4}{a_1} & \frac{\beta\,r_1\,\tau_2}{a_1} & 0 \end{bmatrix}$$

The corresponding eigenvalues are

$\lambda_1 = -\dfrac{a_2 r_1\,\tau_2 c_4 - \alpha\,a_1 d_2 - r_2\,a_1 c_4}{a_1 c_4}$

$\lambda_2 = -\dfrac{\beta\,r_1\,\tau_2 c_4 - d_2 c_3 a_1 + d_1 a_1 c_4}{a_1 c_4}$

$\lambda_3 = 1/2\,\dfrac{-r_1\,(K_1 c_4 \tau_2 - K_1 c_4 + 2\,d_2) + \sqrt{-r_1\,B}}{K_1 c_4}$

$\lambda_4 = -1/2\,\frac{r_1\,(K_1 c_4 \tau_2 - K_1 c_4 + 2\,d_2) + \sqrt{-r_1\,B}}{K_1 c_4}$ Where $B = 2\,{K_1}^2{c_4}^2 r_1\,\tau_2 + 4\,{K_1}^2{c_4}^2 d_2 \tau_2 - {K_1}^2{c_4}^2{\tau_2}^2 - {K_1}^2{c_4}^2 r_1 - 4\,r_1\,K_1 c_4 d_2 \tau_2 + 4\,r_1\,K_1 c_4 d_2 - 4\,r_1\,{d_2}^2$

Hence the equilibrium point (5) is locally asymptotically stable if $\tau_2 = \left(\frac{K_1 c_4 - d_2}{K_1 c_4}\right)$ is locally asymptotically stable if $a_2 r_2 \tau_2 c_4 > a_1(\alpha d_2 + r_2 c_4), \beta r_1 \tau_2 c_4 > a_1(d_2 a_3 - d_1 c_4)$.

Theorem 6.6 Given the linearized system of equations (2.1) is susceptible prey free and infected predator free. In which the equilibrium point $\left\{W = 0, x = \frac{d_1}{a_3}, y = \frac{1}{c_2} r_2 \tau_3, z = 0\right\}$ where $\tau_3 = \left(\frac{K_2 a_3 - d_1}{K_2 a_3}\right)$ is locally asymptotically stable if $c_1 r_2 \tau_3 a_3 > c_2(r_2 a_3 - \alpha d_1), \beta r_2 \tau_3 a_3 > c_2(a_3 d_2 - a_4 d_1)$.

Proof

$$J_6 = \begin{bmatrix} r_1 - \frac{c_1\,r_2\,\tau_3}{c_2} - \frac{\alpha\,d_1}{a_3} & 0 & 0 & 0 \\ \frac{\alpha\,d_1}{a_3} & r_2 - 2\,\frac{r_2\,d_1}{a_3 K_2} - r_2\,\tau_3 & -\frac{d_1 c_2}{a_3} & -\frac{a_2 d_1}{a_3} \\ \frac{r_2\,\tau_3 c_3}{c_2} & \frac{r_2\,\tau_3 a_3}{c_2} & 0 & -\frac{\beta\,r_2\,\tau_3}{c_2} \\ 0 & 0 & 0 & \frac{\beta\,r_2\,\tau_3}{c_2} + \frac{a_4 d_1}{a_3} - d_2 \end{bmatrix}$$

The corresponding eigenvalues are

$\lambda_1 = -\dfrac{c_1\,r_2\,\tau_3 a_3 + \alpha\,d_1 c_2 - r_1\,c_2\,a_3}{c_2\,a_3}$

$\lambda_2 = \dfrac{\beta\,r_2\,\tau_3 a_3 - c_2\,a_3 d_2 + c_2\,a_4 d_1}{c_2\,a_3}$

$\lambda_3 = 1/2 \, \dfrac{-r_2 \, (K_2a_3\tau_3 - K_2a_3 + 2\,d_1) + \sqrt{-r_2\,C}}{K_2a_3}$

$\lambda_4 = -1/2 \, \dfrac{r_2\,\tau_3K_2a_3 - r_2\,K_2a_3 + 2\,d_1r_2 + \sqrt{-r_2\,C}}{K_2a_3}$ where $C = 2\,r_2\,{K_2}^2{a_3}^2\tau_3 + 4\,{K_2}^2{a_3}^2d_1\tau_3 - {K_2}^2{a_3}^2{\tau_3}^2 - r_2\,{K_2}^2{a_3}^2 - 4\,r_2\,K_2a_3d_1\tau_3 + 4\,r_2\,K_2a_3d_1 - 4\,r_2\,{d_1}^2$ hence the equilibrium point (6) is locally asymptotically stable if $c_1r_2\tau_3a_3 > c_2(r_2a_3 - \alpha d_1), \beta r_2\tau_3a_3 > c_2(a_3d_2 - a_4d_1)$.

Theorem 6.7 Given the linearized system of equations (2.1) is susceptible prey free and susceptible predator free. Then the equilibrium point $\left\{W = 0, x = \frac{d_2}{a_4}, y = 0, z = \frac{1}{a_2}r_2\tau_4\right\}$ where $\tau_4 = \left(\frac{K_2a_4 - d_2}{K_2a_4}\right)$ is locally asymptotically stable if $a_1r_2\tau_4a_4 > a_2(r_1a_4 - \alpha d_2), r_2\tau_4\beta a_4 > a_2(d_2a_3 - a_4d_1)$.

Proof

$$J_6 = \begin{bmatrix} r_1 - \frac{a_1r_2\,\tau_4}{a_2} - \frac{\alpha\,d_2}{a_4} & 0 & 0 & 0 \\ \frac{\alpha\,d_2}{a_4} & r_2 - 2\,\frac{r_2\,d_2}{a_4K_2} - r_2\,\tau_4 & -\frac{d_2c_2}{a_4} & -\frac{a_2d_2}{a_4} \\ 0 & 0 & -\frac{\beta\,r_2\,\tau_4}{a_2} + \frac{a_3d_2}{a_4} - d_1 & 0 \\ \frac{r_2\,\tau_4c_4}{a_2} & \frac{r_2\,\tau_4a_4}{a_2} & \frac{\beta\,r_2\,\tau_4}{a_2} & 0 \end{bmatrix}$$

The corresponding eigenvalues are

$\lambda_1 = -\dfrac{a_1r_2\,\tau_4a_4 + \alpha\,d_2a_2 - r_1\,a_2a_4}{a_2a_4}$

$\lambda_2 = -\dfrac{r_2\,\tau_4\beta\,a_4 - d_2a_3a_2 + a_2a_4d_1}{a_2a_4}$

$\lambda_3 = 1/2\,\frac{-r_2\,(K_2a_4\tau_4 - K_2a_4 + 2\,d_2) + \sqrt{-r_2\,M}}{K_2a_4}$

$\lambda_4 = -1/2 \, \dfrac{r_2\,(K_2a_4\tau_4 - K_2a_4 + 2\,d_2) + \sqrt{-r_2\,M}}{K_2a_4}$ where $M = 2\,{K_2}^2{a_4}^2r_2\,\tau_4 + 4\,{K_2}^2{a_4}^2d_2\tau_4 - {K_2}^2{a_4}^2{\tau_4}^2 - {K_2}^2{a_4}^2r_2 - 4\,r_2\,K_2a_4d_2\tau_4 + 4\,r_2\,K_2a_4d_2 - 4\,r_2\,{d_2}^2$ hence the equilibrium point (7) is locally asymptotically stable if $a_1r_2\tau_4a_4 > a_2(r_1a_4 - \alpha d_2), r_2\tau_4\beta a_4 > a_2(d_2a_3 - a_4d_1)$.

Theorem 6.8 Given the linearized system of equations (2.1) is both susceptible–infected prey equilibrium point. Then the equilibrium point $\left\{W = 0, x = 0, y = \frac{d_2}{\beta}, z = -\frac{d_1}{\beta}\right\}$ is locally asymptotically stable if $\beta r_1 > c_1d_2 - a_1d_1, \beta r_2 > (c_2d_2 - a_2d_1)$.

Proof

$$J_7 = \begin{bmatrix} r_1 - \frac{c_1\,d_2}{\beta} + \frac{a_1d_1}{\beta} & 0 & 0 & 0 \\ 0 & r_2 - \frac{d_2c_2}{\beta} + \frac{a_2d_1}{\beta} & 0 & 0 \\ \frac{d_2c_3}{\beta} & \frac{a_3d_2}{\beta} & 0 & -d_2 \\ -\frac{d_1c_4}{\beta} & -\frac{d_1a_4}{\beta} & -d_1 & 0 \end{bmatrix}$$

The corresponding eigenvalues are

$\lambda_1 = \dfrac{\beta\,r_1 - c_1\,d_2 + a_1d_1}{\beta}$ $\lambda_2 = \dfrac{\beta\,r_2 - d_2c_2 + a_2d_1}{\beta}$ $\lambda_3 = \sqrt{d_2d_1}$ $\lambda_4 = -\sqrt{d_2d_1}$ hence the equilibrium point (8) is locally asymptotically stable if $\beta r_1 > c_1d_2 - a_1d_1, \beta r_2 > (c_2d_2 - a_2d_1)$.

Theorem 6.9 Given the linearized system of equations (2.1) is infected predator equilibrium point. Then the equilibrium point $\{W=\vartheta_1, x=\vartheta_2, y=\vartheta_3, z=0\}$ is locally asymptotically stable.

Proof

$$J_8=\begin{bmatrix} m_{11} & -\alpha\vartheta_1 & -\vartheta_1 c_1 & -a_1\vartheta_1 \\ \alpha\vartheta_2 & m_{22} & -\vartheta_2 c_2 & -a_2\vartheta_2 \\ \vartheta_3 c_3 & \vartheta_3 a_3 & m_{33} & -\beta\vartheta_3 \\ 0 & 0 & 0 & m_{44} \end{bmatrix}$$

where $m_{11}=r_1-2\frac{r_1\vartheta_1}{K_1}-c_1\vartheta_3-\alpha\vartheta_2, m_{22}=r_2-2\frac{r_2\vartheta_2}{K_2}-c_2\vartheta_3+\alpha\vartheta_1, m_{33}=a_3\vartheta_2+c_3\vartheta_1-d_1, m_{44}\beta\vartheta_3+a_4\vartheta_2+c_4\vartheta_1-d_2$ The characteristic function are $\Lambda_1(\lambda)=A_0\lambda^4+A_1\lambda^3+A_2\lambda^2+A_3\lambda+A_4$

Where $A_1=(-m_{44}-m_{33}-m_{22}-m_{11})$

$A_2=\alpha^2\vartheta_1\vartheta_2+c_1c_3\vartheta_1\vartheta_3+c_2a_3\vartheta_2\vartheta_3+m_{22}m_{11}+m_{33}m_{11}+m_{44}m_{11}+m_{33}m_{22}+m_{44}m_{22}+m_{44}m_{33}$

$A_3=\alpha c_1a_3\vartheta_1\vartheta_2\vartheta_3-\alpha c_2c_3\vartheta_1\vartheta_2\vartheta_3-\alpha^2m_{33}\vartheta_1\vartheta_2-\alpha^2m_{44}\vartheta_1\vartheta_2-c_1c_3m_{22}\vartheta_1\vartheta_3-c_1c_3m_{44}\vartheta_1\vartheta_3-c_2a_3m_{11}\vartheta_2\vartheta_3-c_2a_3m_{44}\vartheta_2\vartheta_3-m_{11}m_{22}m_{33}-m_{11}m_{22}m_{44}-m_{11}m_{33}m_{44}-m_{22}m_{33}m_{44},$

$A_4=m_{44}(-\alpha c_1a_3\vartheta_1\vartheta_2\vartheta_3+\alpha c_2c_3\vartheta_1\vartheta_2\vartheta_3+\alpha^2m_{33}\vartheta_1\vartheta_2+c_1c_3m_{22}\vartheta_1\vartheta_3+c_2a_3m_{11}\vartheta_2\vartheta_3+m_{11}m_{22}m_{33}).$

By Routh Hurwitzs criterion, all the eigenvalues of J_{10} have negative real parts if $(i)A_0>0$,

$(ii)A_1>0$,

$(iii)A_3>0$,

$(vi)A_1A_2A_3>A_3^2+A_1^2A_4$. We observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (9) if the conditions stated in the theorem holds.

Theorem 6.10 Given the linearized system of equations (2.1) is infected prey equilibrium point. Then the equilibrium point $\{W=\vartheta_4, x=0, y=\vartheta_5, z=\vartheta_6\}$ is locally asymptotically stable.

Proof

$$J_9=\begin{bmatrix} m_{11} & -\alpha\vartheta_4 & -\vartheta_4 c_1 & -a_1\vartheta_4 \\ 0 & m_{22} & 0 & 0 \\ \vartheta_5 c_3 & \vartheta_5 a_3 & m_{33} & -\beta\vartheta_5 \\ \vartheta_6 c_4 & \vartheta_6 a_4 & \beta\vartheta_6 & m_{44} \end{bmatrix}$$

$m_{11}=r_1-2\frac{r_1\vartheta_4}{K_1}-c_1\vartheta_5-a_1\vartheta_6, m_{22}=\alpha\vartheta_4-c_2\vartheta_5-a_2\vartheta_6+r_2, m_{33}=-\beta\vartheta_6+c_3\vartheta_4-d_1, m_{44}=\beta\vartheta_5+c_4\vartheta_4-d_2$. The characteristic function are $\Lambda_2(\lambda)=B_0\lambda^4+B_1\lambda^3+B_2\lambda^2+B_3\lambda+B_4$

Where

$B_0=1,$

$B_1=(-m_{44}-m_{33}-m_{22}-m_{11}),$

$B_2=\beta^2\vartheta_5\vartheta_6+c_1c_3\vartheta_4\vartheta_5+a_1c_4\vartheta_4\vartheta_6+m_{22}m_{11}+m_{33}m_{11}+m_{44}m_{11}+m_{33}m_{22}+m_{44}m_{22}+m_{44}m_{33},$

$B_3=-\beta c_1c_4\vartheta_4\vartheta_5\vartheta_6+\beta a_1c_3\vartheta_4\vartheta_5\vartheta_6-\beta^2m_{11}\vartheta_5\vartheta_6-\beta^2m_{22}\vartheta_5\vartheta_6-c_1c_3m_{22}\vartheta_4\vartheta_5-c_1c_3m_{44}\vartheta_4\vartheta_5-a_1c_4m_{22}\vartheta_4\vartheta_6-a_1c_4m_{33}\vartheta_4\vartheta_6-m_{11}m_{22}m_{33}-m_{11}m_{22}m_{44}-m_{11}m_{33}m_{44}-m_{22}m_{33}m_{44},$

$B_4=\beta c_1c_4m_{22}\vartheta_4\vartheta_5\vartheta_6-\beta a_1c_3m_{22}\vartheta_4\vartheta_5\vartheta_6+\beta^2m_{11}m_{22}\vartheta_5\vartheta_6+c_1c_3m_{22}m_{44}\vartheta_4\vartheta_5+a_1c_4m_{22}m_{33}\vartheta_4\vartheta_6+$

$m_{11}m_{22}m_{33}m_{44}$.
By Routh Hurwitzs criterion, all the eigenvalues of J_9 have negative real parts if $(i)B_0 > 0$,
$(ii)B_1 > 0$,
$(iii)B_3 > 0$,
$(vi)B_1B_2B_3 > B_3^2 + B_1^2B_4$. we observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (10) if the conditions stated in the theorem holds.

Theorem 6.11 Given the linearized system of equations (2.1) is susceptible predator free equilibrium point. Then the equilibrium point $\{W = \vartheta_7, x = \vartheta_8, y = 0, z = \vartheta_9\}$ is locally asymptotically stable.

Proof

$$J_{10} = \begin{bmatrix} m_{11} & -\alpha\,\vartheta_7 & -\vartheta_7 c_1 & -a_1\vartheta_7 \\ \alpha\,\vartheta_8 & m_{22} & -\vartheta_8 c_2 & -a_2\vartheta_8 \\ 0 & 0 & m_{33} & 0 \\ \vartheta_9 c_4 & \vartheta_9 a_4 & \beta\,\vartheta_9 & m_{44} \end{bmatrix}$$

Where
$m_{11} = r_1 - 2\frac{r_1\vartheta_7}{K_1} - a_1\vartheta_9 - \alpha\,\vartheta_8, m_{22} = r_2 - 2\frac{r_2\vartheta_8}{K_2} - a_2\vartheta_9 + \alpha\,\vartheta_7, m_{33} = -\beta\,\vartheta_9 + a_3\vartheta_8 + c_3\vartheta_7 - d_1, m_{44} = a_4\vartheta_8 + c_4\vartheta_7 - d_2$. The characteristic function are $\Lambda_3(\lambda) = C_0\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_3\lambda + C_4$
Where
$C_0 = 1$,
$C_1 = (-m_{44} - m_{33} - m_{22} - m_{11})$,
$C_2 = \alpha^2\vartheta_7\vartheta_8 + a_1c_4\vartheta_7\vartheta_9 + a_2a_4\vartheta_8\vartheta_9 + m_{22}m_{11} + m_{33}m_{11} + m_{44}m_{11} + m_{33}m_{22} + m_{44}m_{22} + m_{44}m_{33}$
$C_3 = \alpha\,a_1a_4\vartheta_7\vartheta_8\vartheta_9 - \alpha\,a_2c_4\vartheta_7\vartheta_8\vartheta_9 - \alpha^2 m_{33}\vartheta_7\vartheta_8 - \alpha^2 m_{44}\vartheta_7\vartheta_8 - a_1c_4m_{22}\vartheta_7\vartheta_9 - a_1c_4m_{33}\vartheta_7\vartheta_9 - a_2a_4m_{11}\vartheta_8\vartheta_9 - a_2a_4m_{33}\vartheta_8\vartheta_9 - m_{11}m_{22}m_{33} - m_{11}m_{22}m_{44} - m_{11}m_{33}m_{44} - m_{22}m_{33}m_{44}$,
$C_4 = -\alpha\,a_1a_4m_{33}\vartheta_7\vartheta_8\vartheta_9 + \alpha\,a_2c_4m_{33}\vartheta_7\vartheta_8\vartheta_9 + \alpha^2 m_{33}m_{44}\vartheta_7\vartheta_8 + a_1c_4m_{22}m_{33}\vartheta_7\vartheta_9 + a_2a_4m_{11}m_{33}\vartheta_8\vartheta_9 + m_{11}m_{22}m_{33}m_{44}$. By Routh Hurwitzs criterion, all the eigenvalues of J_{10} have negative real parts if $(i)C_0 > 0$,
$(ii)C_1 > 0$,
$(iii)C_3 > 0$,
$(vi)C_1C_2C_3 > C_3^2 + C_1^2C_4$. we observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (11) if the conditions stated in the theorem holds.

Theorem 6.12 Given the linearized system of equations (2.1) is susceptible prey free equilibrium point. Then the equilibrium point $\{W = 0, x = \vartheta_{10}, y = \vartheta_{11}, z = \vartheta_{12}\}$ is locally asymptotically stable.

Proof

$$J_{11} = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ \alpha\,\vartheta_{10} & m_{22} & -\vartheta_{10}c_2 & -a_2\vartheta_{10} \\ \vartheta_{11}c_3 & \vartheta_{11}a_3 & m_{33} & -\beta\,\vartheta_{11} \\ \vartheta_{12}c_4 & \vartheta_{12}a_4 & \beta\,\vartheta_{12} & m_{44} \end{bmatrix}$$

where $m_{11} = -\alpha\,\vartheta_{10} - c_1\,\vartheta_{11} - a_1\vartheta_{12} + r_1, m_{22} = r_2 - 2\,\frac{r_2\,\vartheta_{10}}{K_2} - c_2\,\vartheta_{11} - a_2\vartheta_{12}, m_{33} - \beta\,\vartheta_{12} + a_3\vartheta_{10} - d_1, m_{44} = \beta\,\vartheta_{11} + a_4\vartheta_{10} - d_2$. The characteristic function are $\Lambda_4(\lambda) = D_0\lambda^4 + D_1\lambda^3 + D_2\lambda^2 + D_3\lambda + D_4$.
Where
$D_0 = 1,$
$D_1 = (-m_{44} - m_{33} - m_{22} - m_{11})$
$D_2 = \beta^2\vartheta_{11}\vartheta_{12} + c_2\,a_3\vartheta_{10}\vartheta_{11} + a_2a_4\vartheta_{10}\vartheta_{12} + m_{22}m_{11} + m_{33}m_{11} + m_{44}m_{11} + m_{33}m_{22} + m_{44}m_{22} + m_{44}m_{33},$
$D_3 = -\beta\,c_2\,a_4\vartheta_{10}\vartheta_{11}\vartheta_{12} + \beta\,a_2a_3\vartheta_{10}\vartheta_{11}\vartheta_{12} - \beta^2m_{11}\vartheta_{11}\vartheta_{12} - \beta^2m_{22}\vartheta_{11}\vartheta_{12} - c_2\,a_3m_{11}\vartheta_{10}\vartheta_{11} - c_2\,a_3m_{44}\vartheta_{10}\vartheta_{11} - a_2a_4m_{11}\vartheta_{10}\vartheta_{12} - a_2a_4m_{33}\vartheta_{10}\vartheta_{12} - m_{11}m_{22}m_{33} - m_{11}m_{22}m_{44} - m_{11}m_{33}m_{44} - m_{22}m_{33}m_{44}$
$D_4 = \beta\,c_2\,a_4m_{11}\vartheta_{10}\vartheta_{11}\vartheta_{12} - \beta\,a_2a_3m_{11}\vartheta_{10}\vartheta_{11}\vartheta_{12} + \beta^2m_{11}m_{22}\vartheta_{11}\vartheta_{12} + c_2\,a_3m_{11}m_{44}\vartheta_{10}\vartheta_{11} + a_2a_4m_{11}m_{33}\vartheta_{10}\vartheta_{12} + m_{11}m_{22}m_{33}m_{44}$
By Routh Hurwitzs criterion, all the eigenvalues of J_{11} have negative real parts if $(i)D_0 > 0$,
$(ii)D_1 > 0$,
$(iii)D_3 > 0$,
$(vi)D_1D_2D_3 > D_3^2 + D_1^2D_4$. we observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (12) if the conditions stated in the theorem holds.

Theorem 6.13 Given the linearized system of equations (2.1) is nontrivial equilibrium point. Then the equilibrium point $\left\{W = \overline{W}, x = \overline{x}, y = \overline{y}, z = \overline{z}\right\}$ is locally asymptotically stable.

Proof

$$J_{12} = \begin{bmatrix} m_{11} & -\alpha\,\overline{W} & -\overline{W}c_1 & -a_1\overline{W} \\ \alpha\,\overline{x} & m_{22} & -\overline{x}c_2 & -a_2\overline{x} \\ \overline{y}c_3 & \overline{y}a_3 & m_{33} & -\beta\,\overline{y} \\ \overline{z}c_4 & \overline{z}a_4 & \beta\,\overline{z} & m_{44} \end{bmatrix}$$

Where $m_{11} = r_1 - 2\,\frac{r_1\,\overline{W}}{K_1} - c_1\,\overline{y} - a_1\overline{z} - \alpha\,\overline{x}, m_{22} = r_2 - 2\,\frac{r_2\,\overline{x}}{K_2} - c_2\,\overline{y} - a_2\overline{z} + \alpha\,\overline{W}, m_{33} = \overline{W}c_3 - \beta\,\overline{z} + \overline{x}a_3 - d_1, m_{44} = \overline{W}c_4 + \beta\,\overline{y} + \overline{x}a_4 - d_2$. The characteristic function are $\Lambda_4(\lambda) = E_0\lambda^4 + E_1\lambda^3 + E_2\lambda^2 + E_3\lambda + E_4$
where
$E_0 = 1$
$E_1 = (-m_{44} - m_{33} - m_{22} - m_{11})$
$E_2 = \overline{z}c_4a_1\overline{W} + \overline{z}a_4a_2\overline{x} + \beta^2\overline{zy} + m_{44}m_{11} + m_{44}m_{22} + m_{44}m_{33} + \overline{y}a_3\overline{x}c_2 + \overline{y}c_3\overline{W}c_1 + \alpha^2\overline{x}\overline{W} + m_{22}m_{11} + m_{33}m_{11} + m_{33}m_{22}$
$E_3 = -\overline{xz}a_2a_4m_{11} - \overline{xz}a_2a_4m_{33} - \overline{W}\overline{z}a_1c_4m_{22} - \overline{W}\overline{z}a_1c_4m_{33} - \overline{yx}c_2\,a_3m_{44} - \overline{y}\overline{W}c_1\,c_3m_{44} - \overline{yx}c_2\,a_3m_{11} - \overline{y}\overline{W}c_1\,c_3m_{22} + \overline{x}\overline{W}\overline{z}\alpha\,a_1a_4 + \overline{yxz}\beta\,a_2a_3 + \overline{y}\overline{W}\overline{z}\beta\,a_1c_3 + \overline{yx}\overline{W}\alpha\,c_1\,a_3 - \overline{yx}\overline{W}\alpha\,c_2\,c_3 - \overline{y}\overline{W}\overline{z}\beta\,c_1\,c_4 - \overline{x}\overline{W}\overline{z}\alpha\,a_2c_4 - \overline{yxz}\beta\,c_2\,a_4 - \overline{yz}\beta^2m_{11} - \overline{yz}\beta^2m_{22} - \overline{x}\overline{W}\alpha^2m_{44} - \overline{x}\overline{W}\alpha^2m_{33} - m_{11}m_{22}m_{33} - m_{11}m_{22}m_{44} - m_{11}m_{33}m_{44} - m_{22}m_{33}m_{44}$
$E_4 = \overline{y}\overline{W}\overline{z}\beta\,c_1\,c_4m_{22} + \overline{x}\overline{W}\overline{z}\alpha\,a_2c_4m_{33} + \overline{yxz}\beta\,c_2\,a_4m_{11} - \overline{x}\overline{W}\overline{z}\alpha\,a_1a_4m_{33} - \overline{yxz}\beta\,a_2a_3m_{11} - \overline{y}\overline{W}\overline{z}\beta\,a_1c_3m_{22} - \overline{yx}\overline{W}\alpha\,c_1\,a_3m_{44} + \overline{yx}\overline{W}\alpha\,c_2\,c_3m_{44} + \lambda + \overline{yx}\overline{W}\overline{z}c_1\,a_2a_4c_3 + \overline{yx}\overline{W}\overline{z}c_2\,a_1a_3c_4 + \overline{z}c_4\overline{yx}\overline{W}\alpha\,\beta\,c_2 - \overline{z}c_4\overline{yx}\overline{W}c_1\,a_2a_3 - \overline{z}a_4\overline{yx}\overline{W}\alpha\,\beta\,c_1 - \overline{z}a_4\overline{yx}\overline{W}c_2\,a_1c_3 + \beta\,\overline{zyx}\overline{W}\alpha\,a_1a_3 - \beta\,\overline{zyx}\overline{W}\alpha\,a_2c_3 + \overline{yz}\beta^2m_{11}m_{22} + \overline{x}\overline{W}\alpha^2m_{33}m_{44} + \overline{yx}\overline{W}\overline{z}\alpha^2\beta^2 + \overline{xz}a_2a_4m_{11}m_{33} + \overline{W}\overline{z}a_1c_4m_{22}m_{33} + \overline{yx}c_2\,a_3m_{11}m_{44} + \overline{y}\overline{W}c_1\,c_3m_{22}m_{44} + m_{11}m_{22}m_{33}m_{44}$

By Routh Hurwitzs criterion, all the eigenvalues of J_{12} have negative real parts if $(i)E_0 > 0$,
$(ii)E_1 > 0$,
$(iii)E_3 > 0$,
$(vi)E_1E_2E_3 > E_3^2 + E_1^2E_4$. We observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (13) if the conditions stated in the theorem holds.

7. Numerical solution

Numerical solution are equally important beside the analytical findings to verify them. In this section we present computer simulation of different solutions of the system (2.1) using maple 18 programming.

(1) First we take the parameters of the system as $\rho_1 = (\alpha = 10, \beta = 10, c_1 = 0.01, c_2 = 0.06, r_1 = 1, r_2 = 1, K_1 = 1, K_2 = 1, a_1 = .001, a_2 = 0.001, a_3 = 5, a_4 = 2, c_3 = 2, c_4 = 6, d_1 = 0.1, d_2 = 0.1)$. Then the initial conditions satisfied $W(0) = 0, x(0) = 0, y(0) = 0, z(0) = 11$ is infected predator population (see Figure 1).

(2) If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, x(0) = 1, y(0) = 0, z(0) = 0$ is infected prey population (see Figure 2).

(3) If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, x(0) = 0, y(0) = 2, z(0) = 0$ is susceptible predator population (see Figure 3)

(4) If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 1, x(0) = 0, y(0) = 0, z(0) = 0$ is susceptible prey population (see Figure 4).

(5) Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, x(0) = 0, y(0) = 0.5, z(0) = 0.1$ is both susceptible–infected predator population (see Figure 5). That is a periodic solution.

(6) Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 5, x(0) = 0, y(0) = 0, z(0) = 5$ is susceptible prey and infected predator population (see Figure 6).

(7) Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, x(0) = 90, y(0) = 90, z(0) = 0$ is susceptible predator and infected prey population (see Figure 7).

(8) Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 10, x(0) = 10, y(0) = 0, z(0) = 0$ is both susceptible–infected prey population (see Figure 8). That is a periodic solution.

(9) If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0.100, x(0) = 0.100, y(0) = 0.100, z(0) = 0.100$ is both susceptible–infected prey–predator population (see Figure 9).

(10) If we take the parameters of the system as $\rho_2 = (\alpha = 0.02, \beta = 0.04, c_1 = 10, c_2 = 6, r_1 = 20, r_2 = 31, K_1 = 1, K_2 = 1, a_1 = 10, a_2 = 10, a_3 = 15, a_4 = 21, c_3 = 28, c_4 = 16, d_1 = 0.1, d_2 = 0.1)$. Then the initial conditions satisfied $W(0) = 400, x(0) = 400, y(0) = 400, z(0) = 400$ is both interaction of the susceptible–infected prey–predator population (see Figure 10).

(11) If we take the parameters of the system as ρ_2. Then the initial conditions satisfied $W(0) = 80, x(0) = 80, y(0) = 80, z(0) = 80$ for interaction of the both interaction of the susceptible–infected prey–predator population is locally asymptotically stable (see Figure 11).

(12) If we take the parameters of the system as ρ_2. Then the initial conditions satisfied $W(0) = 40, x(0) = 40, y(0) = 40, z(0) = 40$ for interaction of the both interaction of the susceptible–infected prey–predator population is locally asymptotically stable (see Figure 12).

(13) If we take the parameters of the system as ρ_2. Then the initial conditions satisfied $W(0) = 4, x(0) = 4, y(0) = 4, z(0) = 4$ is interaction of the both interaction of the susceptible–infected prey–predator population (see Figure 13).

(14) If we take the parameters of the system as ρ_2. Then the initial conditions satisfied $W(0) = .40, x(0) = .40, y(0) = .40, z(0) = .40$ is interaction of the both interaction of the susceptible–infected prey–predator population (see Figure 14).

(15) If we take the parameters of the system as ρ_2. Then the initial conditions satisfied $W(0) = 0.040, x(0) = 0.040, y(0) = 0.040, z(0) = 0.040$ is interaction of the both interaction of the susceptible–infected prey–predator population (see Figure 15).

8. Conclusion

In this paper, we have studied aim eco-epidemiological model with the assumption that both prey species diseases with susceptible–infection and both predator species diseases susceptible–infection. The present investigation carried out to observe the stability and equilibrium point, boundedness. Moreover, our numerical simulation suggests that in the presence of the environmental fluctuation, the stability analysis is locally asymptotically stable with different equilibrium point.

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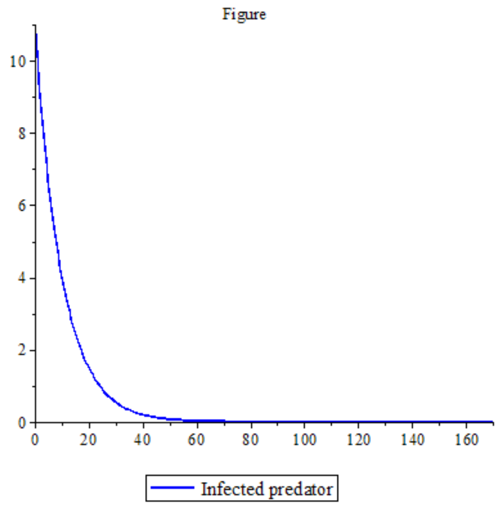


FIGURE 1. The infected predator population.

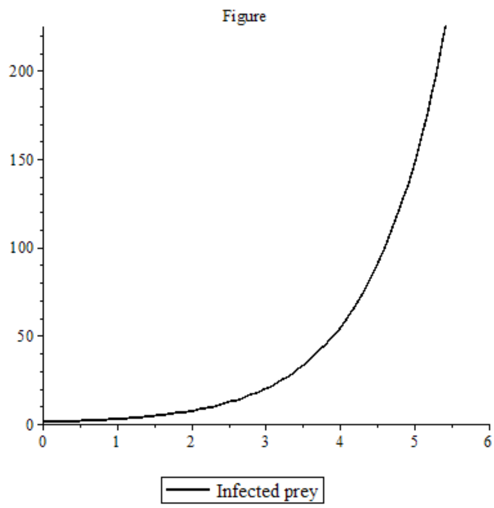


FIGURE 2. The infected prey population.

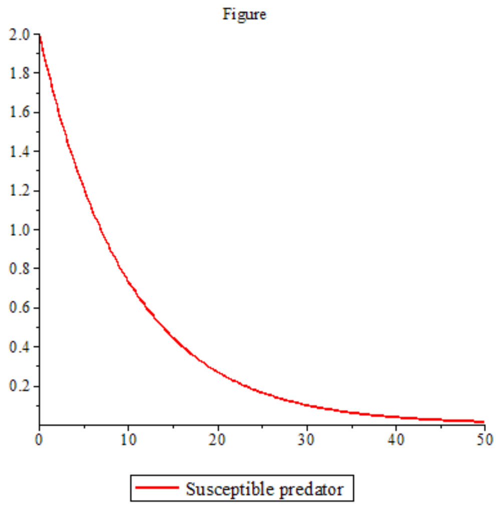
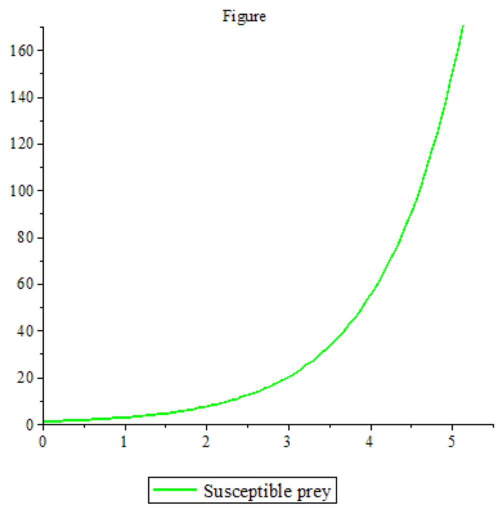


FIGURE 3. The susceptible predator population.



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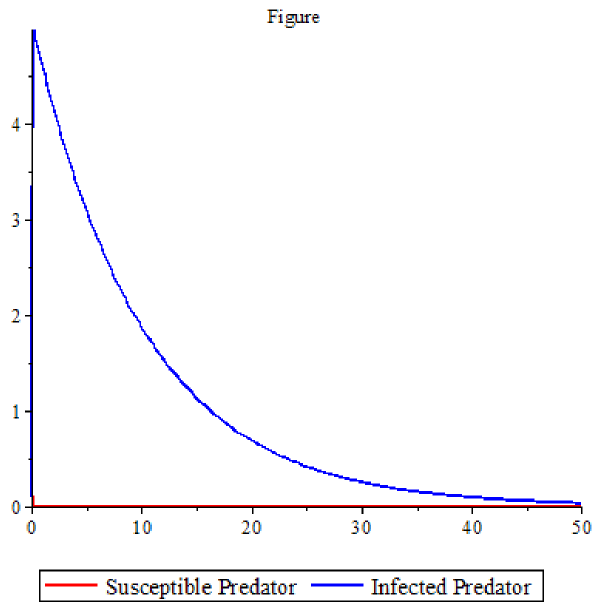

FIGURE 5. The susceptible prey population.

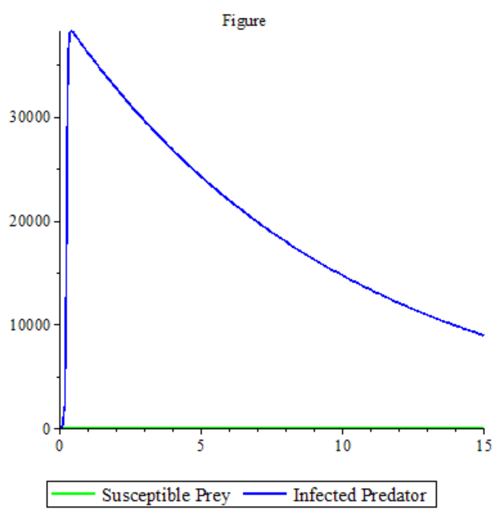

FIGURE 6. Interaction of the susceptible prey and infected predator population.

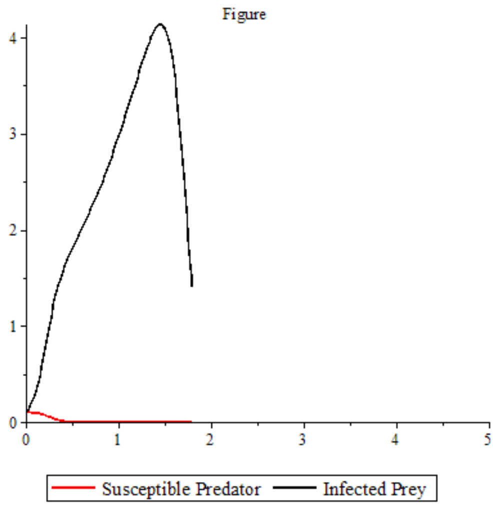

FIGURE 7. Interaction of the susceptible predator and infected prey population.

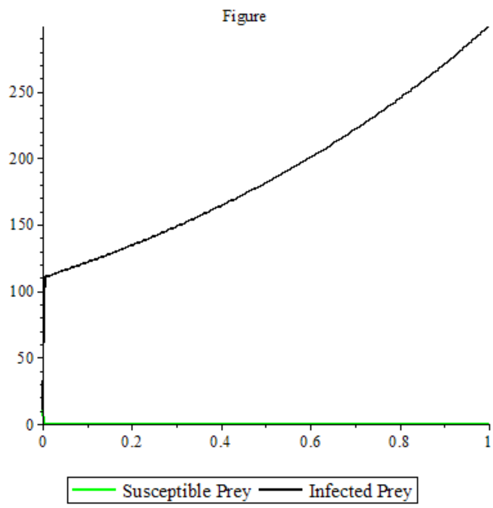

FIGURE 8. Interaction of the susceptible prey and infected prey population.

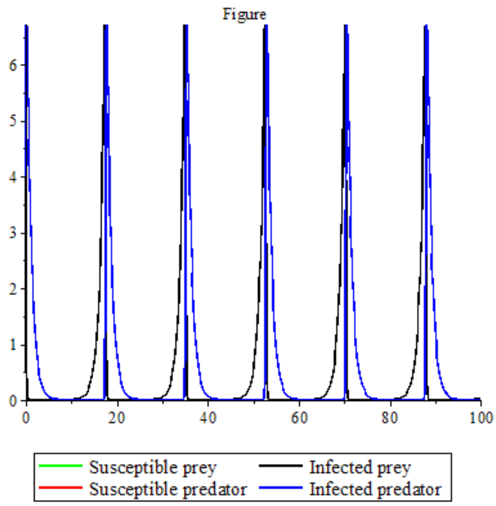

FIGURE 9. Interaction of the both prey predator population.

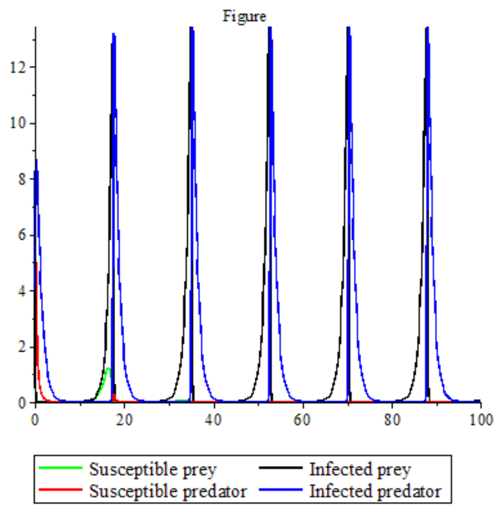

FIGURE 10. Interaction of the both prey predator population.

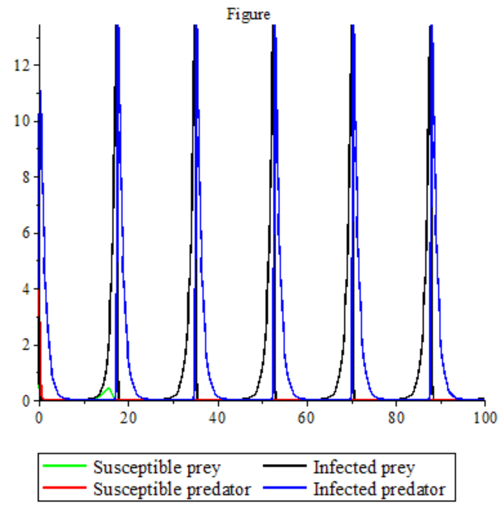

FIGURE 11. Interaction of the both prey predator population.

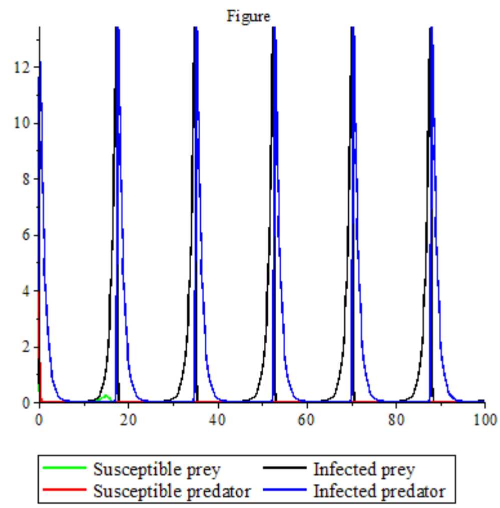

FIGURE 12. Interaction of the both prey predator population.

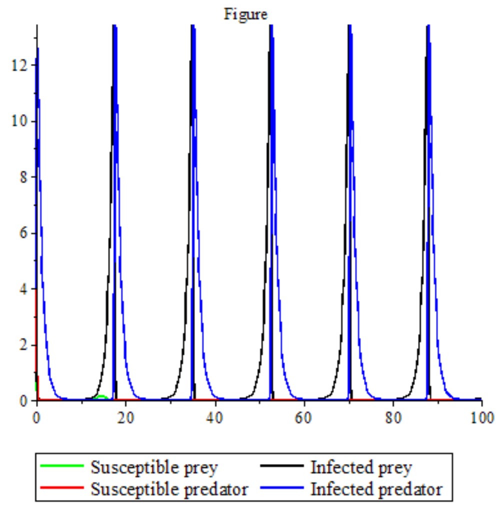

Figure 13. Interaction of the both prey predator population.

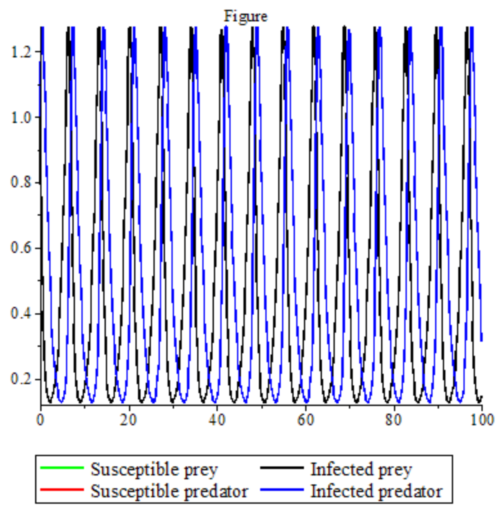

Figure 14. Interaction of the both prey predator population.

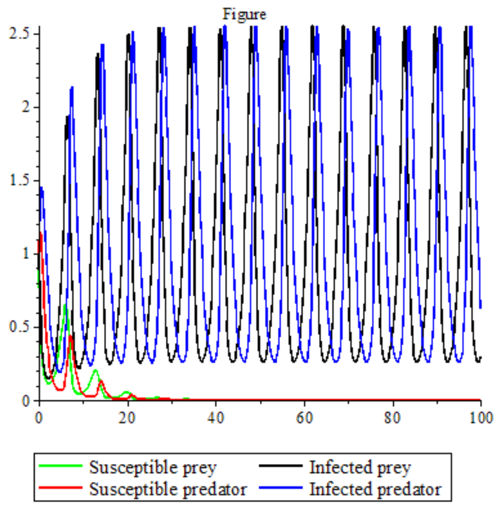

FIGURE 15. Interaction of the both prey predator population.

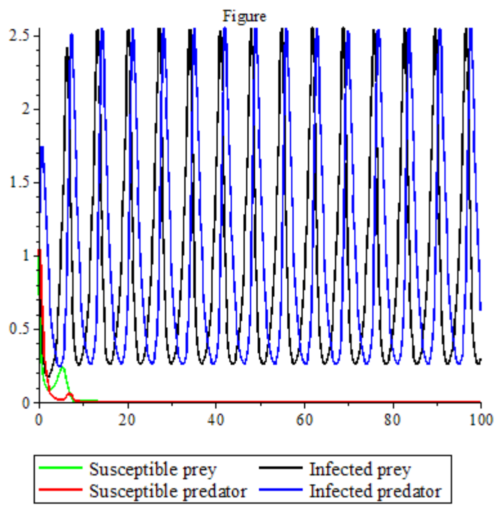

FIGURE 16. Interaction of the both prey predator population.

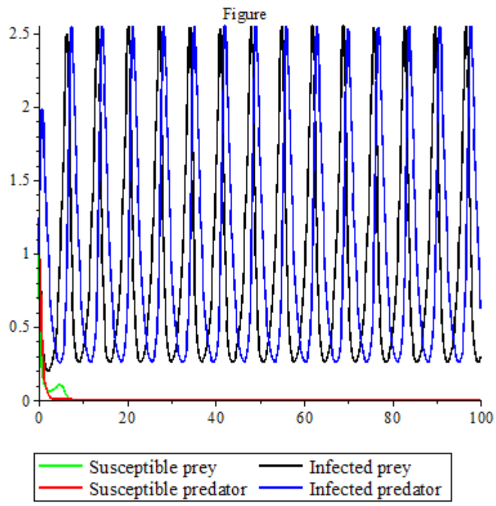

FIGURE 17. Interaction of the both prey predator population.

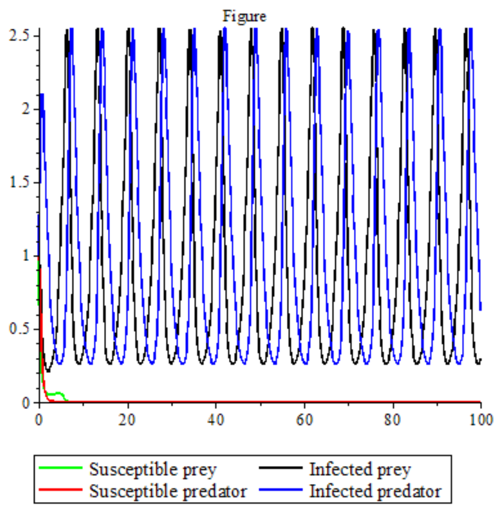

FIGURE 18. Interaction of the both prey predator population.