Bangmod Int. **J.** **M**ath. **&** **C**omp. **S**ci.
Vol. 3, No. 1-2, 2017; Pages 123 - 137
http://bangmod-jmcs.kmutt.ac.th/
ISSN : 2408-154X

Best proximity point results for Suzuki type generalized $(\psi - \phi)-$ weak proximal contraction mapping

K. K. M. Sarma[1] **and Yohannes Gebru Aemro**[2]
[1, 2,] *Department of Mathematics, Andhra University, Visakhapatnam-530 003, India,* [2] *Department of Mathematics, Wolkite university, Wolkite, Ethiopia.*
[1] *e-mail: sarmakmkandala@yahoo.in,* [2] *e-mail:yohannnesgebru2005@gmail.com*

[1]*Corresponding author.*

Abstract In this paper, we introduce a new Suzuki type generalized weak proximal contraction mapping and prove the existence of the best proximity points for such mappings in a complete metric space. We provide examples to illustrate our result. Our result extends some of the results in the literature.

MSC: Primary: 47H10; Secondary: 54H25, 11J83.
Keywords: Best Proximty Point, Suzuki type generalized weak proximal contraction , (ψ, ϕ)-contractive conditions.

Submission date: 11 November 2017 / Acceptance date: 26 December 2017 /Available online 31 December 2017

1. Introduction

Many problems can be formulated as equations of the form $Tx = x$, where T is a self-mapping with some suitable domains. From the fact that fixed point theory plays an important role in furnishing a uniform treatment to solve various equations of the form $Tx = x$, however, in the case that T is non-self-mapping, the given equation does not necessarily have a fixed point. In such case, it is worthy to determine an approximate solution x such that the error $d(x, Tx)$ is minimum. This is the idea behind best approximation theory. A classical best approximation theorem was introduced by Fan[2]; that is, if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterwards, several authors, including Prolla [7], Reich [9], and Sehgal and Singh [12] , have derived extensions of Fans Theorem in many directions. A number of authors have improved, generalized and extended this basic result either by

Published by **T**heoretical **and** **C**omputational **S**cience Center **(TaCS)**,
King Mongkut's University of Technology Thonburi **(KMUTT)**

Bangmod-JMCS
Available online @ http://bangmod-jmcs.kmutt.ac.th/

defining a new contractive mapping in the context of a complete metric space or extends best proximity results from fixed point theory(see [3],[6],[10][11]).

Recently, Shyam et al.[13], introduced a new class of contraction mappings called generalized weak contractions for self mappings and in their work they extend Suzuki theorem [14].

Let Ψ denote the class of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ψ is monotone non-decreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0 \iff t = 0$,

and let Φ is denote the class of all functions
$\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ϕ is lower semi-continuous function ;
(b) $\phi(t) = 0 \iff t = 0$.

The following theorem is proved by Shyam et al. in [13].

Theorem 1.1. *[13] Let X be a complete metric space. $T : X \to X$ be self map such that for every $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \le d(x, y) \text{ implies } \psi(d(Tx, Ty)) \le \psi(M_g(Tx, Ty)) - \phi(M_g(Tx, Ty)), \tag{1.1}$$

where $\psi \in \Psi, \phi \in \Phi$ and $M_g(Tx, Ty) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. Then T has a unique fixed point.

In this direction for more details we refer the reader to [13].

In this paper, we extend the concept of generalized weak contraction type mappings to the case of non self mappings. In particular we study the existence of best proximity points for generalized Suzuki type $(\psi - \phi)$- weak proximal contraction mappings. Further we present several consequences of our obtained results.

2. Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). We use the following notations:
We write
$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$,
$A_0 = \{a \in A : d(a, b) = d(A, B)$ for some $b \in B\}$;
$B_0 = \{b \in A : d(a, b) = d(A, B)$ for some $a \in A\}$.

Remark 2.1. If $A = B$ then $A_0 = A = B = B_0$.

Definition 2.2. An element $x^* \in A$ is said to be a best proximity point of the non-self mapping $T : A \to B$ if it satisfies the condition that $d(x^*, Tx^*) = d(A, B)$.

We denote the set of all best proximity points of T by $P_T(A)$, that is:

$$P_T(A) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

In [4], J. Hamzehnejadi, R. Lashkaripour introduced a property konown as $RJ-$ property and proved best proximity results. The property is as follows.

Definition 2.3. [4] Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. We say that T has $RJ-$ property if for any sequence $\{x_n\} \subset A$,

$$\left.\begin{array}{l}\lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B)\\ \lim_{n\to\infty} x_n = x\end{array}\right\} \implies x \in A_0.$$

Remark 2.4. [4] Every continuous mapping has $RJ-$ property.

We refer the reader to [4] for some more details.

The following lemma is proved by G.V.R. Babu and P.D.Sailaja in [1].

Lemma 2.5. *[1] Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

i): $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$;
ii): $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$;
iii): $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.

Remark 2.6. By using the hypotheses of Lemma 2.5 and triangular inequality it can be shown that $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \epsilon$, $\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon$ and $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon$.

In this paper, we propose new types of Suzuki type proximal maps to prove best proximity point results.

3. Main results

We now introduce the following definition.

Definition 3.1. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a generalized Suzuki type $(\psi - \phi)$- weak proximal contraction if there exists $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y, u, v \in A$,

$$\left.\begin{array}{l}\frac{1}{2}d(x, u) \leq d(x, y)\\ d(u, Tx) = d(A, B)\\ d(v, Ty) = d(A, B)\end{array}\right\} \implies \tag{3.1}$$

$$\psi(d(u, v)) \leq \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v)),$$

where
$M_T(x, y, u, v) = \max\{d(x, y), d(x, u), d(y, v), \frac{d(x,v)+d(y,u)}{2}\}$.

Now we prove the following theorem, which extends, improves and generalizes some of the results in the literature on best proximity points.

Theorem 3.2. *Let A and B be two nonempty, closed subsets of a complete metric space (X, d). Let $T : A \to B$ be a mapping. Suppose that the following conditions are satisfied:*

i): *T is a generalized Suzuki type $(\psi - \phi)$-weak proximal contraction mapping;*
ii): *$T(A_0) \subseteq B_0$;*
iii): *T has $RJ-$ property;*
iv): *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then T has a unique best proximity point.

Proof. Let $x_1, x_0 \in A$ be such that $d(x_1, Tx_0) = d(A, B)$. Then by the definition of A_0 we have that $x_1 \in A_0$. Since $x_1 \in A_0$ and $T(A_0) \subseteq B_0$ there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Again $x_2 \in A_0$ and $T(A_0) \subseteq B_0$ imply that there exists x_3 in A_0 such that $d(x_3, Tx_2) = d(A, B)$.
On continuing this process, by induction, we construct a sequence $\{x_n\} \subseteq A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Now for any $n \in \mathbb{N}$ we have

$$\begin{aligned} &\tfrac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n);\\ &d(x_n, Tx_{n-1}) = d(A, B);\\ &d(x_{n+1}, Tx_n) = d(A, B). \end{aligned}$$

Since T is a generalized Suzuki type $(\psi - \phi)$-weak proximal contraction mapping we have that

$$\psi(d(x_n, x_{n+1})) \leq \psi(M_T(x_{n-1}, x_n, x_n, x_{n+1})) - \phi(M_T(x_{n-1}, x_n, x_n, x_{n+1})), \tag{3.3}$$

where

$$\begin{aligned} M_T(x_{n-1}, x_n, x_n, x_{n+1}) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\}, \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\}. \end{aligned} \tag{3.4}$$

Suppose that there exists $n_0 \in N$ such that $x_{n_0+1} = x_{n_0}$.

Then by (3.2), we have

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

This implies x_{n_0} is best proximity point of T.
Suppose $x_{n+1} \neq x_n$ for any $n \in \mathbb{N}$.
Now from triangular inequality we have

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

So that

$$\frac{d(x_{n-1}, x_{n+1})}{2} \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{3.5}$$

From (3.4) and (3.5) we have $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$, for any $n \in \mathbb{N}$.

If $M_T(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_n, x_{n+1})$ for some $n \in \mathbb{N}$, applying (3.3), we deduce that

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})).$$

This implies $\phi(d(x_n, x_{n+1})) = 0$. From the property of ϕ, we get that $d(x_n, x_{n+1}) = 0$. Consequently $x_n = x_{n+1}$, which is not true. Thus, we conclude that

$$M_T(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Now from (3.3) and (3.6), for all $n \in \mathbb{N}$ we get

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(d(x_{n-1}, x_n)) - \phi((d_{n-1}, x_n)), \\ &< \psi(d(x_{n-1}, x_n)). \end{aligned} \tag{3.7}$$

Since ψ is nondecreasing from the above inequality we get that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}. \tag{3.8}$$

Therefore the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nondecreasing. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

By the lower semi continuity of ϕ, we have

$$\phi(r) \leq \liminf_{n\to\infty} \phi(d(x_n, x_{n-1})). \tag{3.9}$$

Taking limit superior as $n \to \infty$ in (3.7) and using 3.9, we obtain

$$\psi(r) \leq \psi(r) - \phi(r).$$

Hence $\phi(r) = 0$. This implies $r = 0$. So we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0.$$

Now we shall prove that $\{x_n\}$ is a cauchy sequence in (X, d).

Suppose on the contrary that $\{x_n\}$ is not Cauchy. Then by Lemma 2.5 , there exists an $\epsilon > 0$ for which we can find sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and the identities (i)-(iii) of Lemma 2.5 and Remark 2.6 are holds. Hence for any $k \in \mathbb{N}$ we have

$$d(x_{m_k}, x_{n_k}) > d(x_{m_k-1}, x_{n_k}). \tag{3.10}$$

Furthermore from (3.8) and triangular inequality of metric we have

$$\begin{aligned} d(x_{m_k}, x_{m_k+1}) < d(x_{m_k-1}, x_{m_k}) &\leq d(x_{m_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ &\leq 2d(x_{n_k}, x_{m_k}). \end{aligned}$$

Thus we have, $\frac{1}{2} d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k})$.

Moreover we have

$$\begin{aligned} d(x_{m_k+1}, Tx_{m_k}) &= d(A, B); \\ d(x_{n_k+1}, Tx_{n_k}) &= d(A, B). \end{aligned}$$

Since T is a generalized $(\psi - \phi)$-weak proximal contraction type mapping, from the above results, we can conclude that

$$\psi(d(x_{m_k+1}, x_{n_k+1}) \leq \psi(M_T(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1})) - \phi(M_T(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1})), \tag{3.11}$$

where

$$M_T(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) = \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})}{2}\}.$$

(3.12)

As $k \to \infty$ in 3.12, we get that

$$\lim_{k\to\infty} M_T(x_{m_k}, x_{n_k}, x_{m_k+1}, x_{n_k+1}) = \epsilon. \tag{3.13}$$

By taking limit superior in (3.11) and using (3.13), we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

This implies $\phi(\epsilon) = 0$. Hence $\epsilon = 0$, which is not true. Thus $\{x_n\}$ is a Cauchy sequence in A.

Since X is complete and A is a closed subset of X there exists z in A such that $x_n \to z$. Now (3.2) and $RJ-$ property of T implies that $z \in A_0$.

Suppose that there exists $n_0 \in N$ such that

$$\frac{1}{2}d(x_{2n_0}, x_{2n_0+1}) > d(x_{2n_0}, z) \text{ and } \frac{1}{2}d(x_{2n_0+1}, x_{2n_0+2}) > d(x_{2n_0+1}, z).$$

Then we have

$$\begin{aligned} d(x_{2n_0}, x_{2n_0+1}) &\leq d(x_{2n_0}, z) + d(z, x_{2n_0+1}) \\ &< \frac{1}{2}d(x_{2n_0}, x_{2n_0+1} + \frac{1}{2}d(x_{2n_0+1}, x_{2n_0+2}) \\ &< \frac{1}{2}(d(x_{2n_0}, x_{2n_0+1}) + d(x_{2n_0}, x_{2n_0+1})) \text{ (as } \{d(x_n, x_{n+1})\} \text{ is non-increasing)} \\ &= d(x_{2n_0}, x_{2n_0+1}). \end{aligned}$$

This is a contradiction. Therefore for any $n \in \mathbb{N}$ either

$$\frac{1}{2}d(x_{2n}, x_{2n+1}) < d(x_{2n}, z) \text{ or } \frac{1}{2}d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, z). \tag{3.14}$$

Thus there exists a subsequence $\{n_k\}$ of the sequence $\{n\}$ such that

$$\frac{1}{2}d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, z) \text{ for every } k \in \mathbb{N}.$$

Since $z \in A_0$ and $T(A_0) \subseteq B_0$, there exists $w \in A$ such that $d(w, Tz) = d(A, B)$. We shall show that $z = w$.

We have that

$$\begin{aligned} \frac{1}{2}d(x_{n_k}, x_{n_k+1}) &\leq d(x_{n_k}, z); \\ d(x_{n_k+1}, Tx_{n_k}) &= d(A, B); \\ d(w, Tz) &= d(A, B). \end{aligned}$$

By definition (3.1), we have

$$\psi(d(x_{n_k+1}, w)) \leq \psi(M_T(x_{n_k}, z, x_{n_k+1}, w)) - \phi(M_T(x_{n_k}, z, x_{n_k+1}, w)), \tag{3.15}$$

where

$$M_T(x_{n_k}, z, x_{n_k+1}, w) = \max\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), d(z, w), \frac{d(x_{n_k}, w) + d(z, x_{n_k+1})}{2}\}. \tag{3.16}$$

Since $\lim_{k\to\infty} d(x_{n_k}, z) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$,

$$\lim_{k\to\infty} M_T(x_{n_k}, z, x_{n_k+1}, w) = d(z, w).$$

Taking limit superior in (3.15), we obtain

$$\psi(d(z,w)) \leq \psi(d(z,w)) - \phi(d(z,w)).$$

This implies $\phi(d(z,w)) = 0$. By the property of ϕ, $d(z,w) = 0$. Consequently $z = w$. Therefore $d(z,Tz) = d(A,B)$. Hence z is a best proximity point of T.
To show the uniqueness, let u, v be best proximity points of T.
Here we have

$$\begin{array}{rcl} \frac{1}{2}d(u,v) & \leq & d(u,v) \\ d(u,Tu) & = & d(A,B) \\ d(v,Tv) & = & d(A,B) \end{array}$$

By Definition 3.1 we have

$$\psi(d(u,v)) \leq \psi(M_T(u,v,u,v)) - \phi(M_T(u,v,u,v)), \tag{3.17}$$

where

$$M_T(u,v,u,v) = \max\{d(u,v), d(u,u), d(v,v), \frac{d(u,v)+d(v,u)}{2}\} = d(u,v).$$

Thus from (3.17) we get

$$\psi(d(u,v)) \leq \psi(d(u,v)) - \phi(d(u,v)).$$

This implies that $\phi(d(u,v)) = 0$. From the property of ϕ, we get that $d(u,v) = 0$. Therefore $u = v$. Hence T has a unique best proximity point.

∎

Remark 3.3. Observe that the condition (iv) of Theorem 3.2 is equivalent to that $A_0 \neq \emptyset$.

Now we draw some corollaries to our theorem.

Since every Continuous mapping has $RJ-$ property we deduce the following corollary.

Corollary 3.4. *Let A and B be two nonempty closed subsets of a complete metric space (X,d). Let $T : A \to B$ be a continuous mapping such that the following conditions are satisfied:*

i): *T is a generalized $(\psi - \phi)-$ weak proximal contraction type mapping;*
ii): *$T(A_0) \subseteq B_0$;*
iii): *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$.*

Then there exists a unique $x^* \in A_0$ such that $d(x^*, Tx^*) = d(A,B)$.

If we take $\psi(t) = t$ and $\phi(t) = (1-k)t$, where $0 \leq k < 1$ in Theorem 3.2 we get the following corollary.

Corollary 3.5. *Let A and B be two nonempty closed subsets of a complete metric space (X,d). Let $T : A \to B$ be a continuous mapping. Suppose that for all $x, y, u, v \in A$ the following conditions are satisfied:*

i):

$$\left.\begin{array}{l}\frac{1}{2}d(x,u)\leq d(x,y)\\ d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow d(u,v)\leq kM_T(x,y,u,v),$$

where

$$M_T(x,y,u,v)=\max\{d(x,y),d(x,u),d(y,v),\frac{d(x,v)+d(y,u)}{2}\};$$

ii): $T(A_0)\subseteq B_0$;
iii): *there exist* $x_0, x_1 \in A$ *such that* $d(x_1, Tx_0) = d(A, B)$.

Then there exists a unique $x^* \in A_0$ *such that* $d(x^*, Tx^*) = d(A, B)$.

A direct consequence of Theorem 3.2 is the following corollary.

Corollary 3.6. *Let* A *and* B *be two nonempty closed subsets of a complete metric space* (X, d). *Let* $T : A \to B$ *be a continuous mapping. Suppose that for all* $x, y, u, v \in A$ *the following conditions are satisfied:*

i):

$$\left.\begin{array}{l}d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\}\Longrightarrow \tag{3.18}$$

$$\psi(d(u,v))\leq\psi(M_T(x,y,u,v))-\phi(M_T(x,y,u,v)),$$

where $M_T(x,y,u,v)=\max\{d(x,y),d(x,u),d(y,v),\frac{d(x,v)+d(y,u)}{2}\}$;
ii): $T(A_0)\subseteq B_0$;
iii): *there exist* $x_0, x_1 \in A$ *such that* $d(x_1, Tx_0) = d(A, B)$.

Then there exists a unique $x^* \in A_0$ *such that* $d(x^*, Tx^*) = d(A, B)$.

The following example shows that Theorem 3.2 generalizes Corollary 3.6. Further, it is interesting to note that the map T of Example 3.7 does not satisfy the hypotheses of the Corollary 3.6.

Example 3.7. Let $X = R^3$, $d : X \times X \to R$ defined by

$$d((x_1,x_2,x_3),(y_1,y_2,y_3))=|x_1-y_1|+|x_2-y_2|+|x_3-y_3|,$$

$A = \{(0,0,0),(0,4,0),(4,0,0),(4,5,0),(5,4,0)\}$,
$B = \{(0,0,1),(0,4,1),(4,0,1),(4,5,1),(5,4,1)\}$.

We define $T : A \to B$ by

$$T(x_1,x_2,0)=\begin{cases}(x_1,0,1), & \text{if } x_1\leq x_2;\\ (0,x_2,1), & \text{if } x_1> x_2.\end{cases}$$

Clearly $d(A, B) = 1$, $A_0 = A$, $B_0 = \{(0,0,1),(0,4,1),(4,0,1),(4,5,1),(5,4,1)\}$. Notice that $T(A_0) \subseteq B_0$ and T is continuous. Now we choose, $x_1 = (4,0,0)$ and $x_0 = (4,5,0)$, then $d(x_1, Tx_0) = 1$. Now we define functions $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$ and $\phi(t) = \frac{t}{7}$.
We now show that T is a generalized Suzuki type (ψ, ϕ)-weak proximal contraction. For

this regard let $x = (x_1, x_2, 0)$, $y = (y_1, y_2, 0)$, $u = (u_1, u_2, 0)$ and $v = (v_1, v_2, 0) \in A$ such that

$$\frac{1}{2}d(x,u) \leq d(x,y),$$
$$d(u,Tx) = d(A,B),$$
$$d(v,Ty) = d(A,B).$$

Now $v = (v_1, v_2, 0) = \begin{cases} (y_1, 0, 0), & \text{if } y_1 \leq y_2 \\ (0, y_2, 0), & \text{if } y_1 > y_2 \end{cases}$

Now we have to verify the inequality of Definition 3.1 for the following cases.

Case i) Let $x = (0,0,0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(0,0,0)) = 1$ we get $u_1 = 0, u_2 = 0$. Thus, $u = (0,0,0)$. Further more $\frac{1}{2}d(x,u) \leq d(x,y)$ implies that $0 \leq d(x,y)$.
Sub-case i) if $y_1 \leq y_2$ then $(v_1, v_2, 0) = (y_1, 0, 0)$. Now $d(u,v) = y_1$ and $M_T(x,y,u,v) = max\{y_1 + y_2, 0, y_2, \frac{2y_1+y_2}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leq y_2$ we can easily observe that $y_1 \leq \frac{6}{7}y_2 \leq \frac{6}{7}M_T(x,y,u,v)$. Thus $\psi(d(u,v)) = d(u,v) = y_1 \leq \frac{6}{7}M_T(x,y,u,v) = M_T(x,y,u,v) - \frac{1}{7}M_T(x,y,u,v) = \psi(M_T(x,y,u,v)) - \phi(M_T(x,y,u,v))$
Sub-case ii) if $y_1 > y_2$ then $(v_1, v_2, 0) = (0, y_2, 0)$. Now $d(u,v) = y_2$ and $M_T(x,y,u,v) = max\{y_1 + y_2, 0, y_2, \frac{y_1+2y_2}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 > y_2$ we can easily observe that $y_2 \leq \frac{6}{7}y_1 \leq \frac{6}{7}M_T(x,y,u,v)$. Thus $\psi(d(u,v)) = d(u,v) = y_2 \leq \frac{6}{7}M_T(x,y,u,v) = M_T(x,y,u,v) - \frac{1}{7}M_T(x,y,u,v) = \psi(M_T(x,y,u,v)) - \phi(M_T(x,y,u,v))$.
Therefore in this case (3.1) is satisfied.

Case ii) Let $x = (0,4,0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(0,4,0)) = 1$ we get $u_1 = 0, u_2 = 0$. Thus, $u = (0,0,0)$. Moreover,

$$\frac{1}{2}d(x,u) \leq d(x,y) \text{ implies that } 2 \leq d(x,y), \text{ i.e, } 2 \leq y_1 + |y_2 - 4|. \tag{3.19}$$

Sub- case i if $y_1 \leq y_2$ then $v = (y_1, 0, 0)$.
For $y = (0,4,0)$, (3.19) not satisfied, so we don't need to check (3.1) for $y = (0,4,0)$. Now $d(u,v) = y_1$ and $M_T(x,y,u,v) = max\{y_1 + |y_2 - 4|, 4, y_2, \frac{2y_1+y_2+4}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leq y_2$ we can easily observe that $y_1 \leq \frac{6}{7}y_2 \leq \frac{6}{7}M_T(x,y,u,v)$. Thus $\psi(d(u,v)) = d(u,v) = y_1 \leq \frac{6}{7}M_T(x,y,u,v) = M_T(x,y,u,v) - \frac{1}{7}M_T(x,y,u,v) = \psi(M_T(x,y,u,v)) - \phi(M_T(x,y,u,v))$.
Sub- case ii) if $y_1 > y_2$ then $v = (v_1, v_2, 0) = (0, y_2, 0)$. Now $d(u,v) = y_2$ and $M_T(x,y,u,v) = max\{y_1 + |y_2 - 4|, 4, y_1, \frac{y_1+|y_2-4|+y_2}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 > y_2$ we can easily observe that $y_2 \leq \frac{6}{7}y_1 \leq \frac{6}{7}M_T(x,y,u,v)$. Thus $\psi(d(u,v)) = d(u,v) = y_2 \leq \frac{6}{7}M_T(x,y,u,v) = M_T(x,y,u,v) - \frac{1}{7}M_T(x,y,u,v) = \psi(M_T(x,y,u,v)) - \phi(M_T(x,y,u,v))$.

Therefore in this case (3.1) is satisfied.

Case iii) Let $x = (4,0,0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(4,0,0)) = 1$ we get $u_1 = 0, u_2 = 0$. Thus, $u = (0,0,0)$. Again

$$\frac{1}{2}d(x,u) \leq d(x,y) \text{ implies that } 2 \leq d(x,y), \text{ i.e, } 2 \leq |y_1 - 4| + y_2. \tag{3.20}$$

Sub case i) if $y_1 \leq y_2$ then $v = (y_1, 0, 0)$. Now $d(u, v) = y_1$ and $M_T(x, y, u, v) = max\{|y_1 - 4| + y_2, 4, y_2, \frac{y_1+|y_1-4|+y_2}{2}\}$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leq y_2$ we can easily observe that $y_1 \leq \frac{6}{7}y_2 \leq \frac{6}{7}M_T(x, y, u, v)$. Thus $\psi(d(u, v)) = d(u, v) = y_1 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$.
Sub case ii) if $y_1 > y_2$ then $v = (v_1, v_2, 0) = (0, y_2, 0)$.
Now $d(u, v) = y_2$ and $M_T(x, y, u, v) = max\{|y_1 - 4| + y_2, 4, y_1, \frac{y_1+2y_2+4}{2}\}$.
For $y = (4, 0, 0)$, (3.20) not satisfied, so we don't need to check (3.1) for $y = (4, 0, 0)$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 > y_2$ we can easily observe that $y_2 \leq \frac{6}{7}y_1 \leq \frac{6}{7}M_T(x, y, u, v)$. Thus $\psi(d(u, v)) = d(u, v) = y_2 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$.

Therefore in this case (3.1) is satisfied.

Case iv) Let $x = (4, 5, 0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(4, 5, 0)) = 1$ we get $u_1 = 4, u_2 = 0$. Thus, $u = (4, 0, 0)$. Now

$$\frac{1}{2}d(x, u) \leq d(x, y) \text{ implies that } \frac{5}{2} \leq d(x, y), \ i.e, \ \frac{5}{2} \leq |y_1 - 4| + |y_2 - 5|. \quad (3.21)$$

Sub case i) if $y_1 \leq y_2$ then $v = (y_1, 0, 0)$. Now $d(u, v) = |y_1 - 4|$ and $M_T(x, y, u, v) = max\{|y_1-4|+|y_2-5|, 5, y_2, \frac{2|y_1-4|+y_2+5}{2}\}$. For $y = (4, 5, 0)$, (3.21) not satisfied, so we don't need to check (3.1) for $y = (4, 5, 0)$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leq y_2$ we can easily observe that $|y_1-4| \leq \frac{6}{7}\cdot 5 \leq \frac{6}{7}M_T(x, y, u, v)$. Thus $\psi(d(u, v)) = d(u, v) = |y_1-4| \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$.
Sub case ii) if $y_1 > y_2$ then $v = (v_1, v_2, 0) = (0, y_2, 0)$. Now $d(u, v) = y_2 + 4$ and $M_T(x, y, u, v) = max\{|y_1 - 4| + |y_2 - 5|, 5, y_1, \frac{4+|y_1-4|+|y_2-5|+y_2}{2}\}$. For $y = (5, 4, 0)$, (3.21) not satisfied, so we don't need to check (3.1) for $y = (5, 4, 0)$. Here for all $y = (y_1, y_2, 0) \in A$, $y_1 > y_2$ and $(y_1, y_2, 0) \neq (5, 4, 0)$ we can easily observe that $4+y_2 \leq \frac{6}{7}cdot5 \leq \frac{6}{7}M_T(x, y, u, v)$. Thus $\psi(d(u, v)) = d(u, v) = y_2+4 \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$.

Therefore in this case (3.1) is satisfied.

Case v) Let $x = (5, 4, 0)$ and $y = (y_1, y_2, 0)$. From $d((u_1, u_2, 0), T(5, 4, 0)) = 1$ we get $u_1 = 0, u_2 = 4$. Thus, $u = (0, 4, 0)$. Now

$$\frac{1}{2}d(x, u) \leq d(x, y) \text{ implies that } \frac{5}{2} \leq d(x, y), \ i.e, \ \frac{5}{2} \leq |y_1 - 5| + |y_2 - 4|. \quad (3.22)$$

Sub case i) if $y_1 \leq y_2$ then $v = (y_1, 0, 0)$. Now $d(u, v) = y_1 + 4$ and $M_T(x, y, u, v) = max\{|y_1 - 5| + |y_2 - 4|, 5, y_2, \frac{|y_1-5|+|y_2-4|+y_1+4}{2}\}$. For $y = (4, 5, 0)$, (3.22) not satisfied, so we don't need to check (3.1) for $y = (4, 5, 0)$. Here for all $y = (y_1, y_2, 0) \in A$ and $y_1 \leq y_2$ we can easily observe that $y_1 + 4 \leq \frac{6}{7} \cdot 5 \leq \frac{6}{7}M_T(x, y, u, v)$. Thus $\psi(d(u, v)) = d(u, v) = |y_1 - 4| \leq \frac{6}{7}M_T(x, y, u, v) = M_T(x, y, u, v) - \frac{1}{7}M_T(x, y, u, v) = \psi(M_T(x, y, u, v)) - \phi(M_T(x, y, u, v))$.
Sub case ii) if $y_1 > y_2$ then $v = (v_1, v_2, 0) = (0, y_2, 0)$. Now $d(u, v) = |y_2 - 4|$ and $M_T(x, y, u, v) = max\{|y_1-5|+|y_2-4|, 5, y_1, \frac{y_1+2|y_2-4|+5}{2}\}$. For $y = (5, 4, 0)$, (3.22) not satisfied, so we don't need to check (3.1) for $y = (5, 4, 0)$. Here for all $y = (y_1, y_2, 0) \in A$, $y_1 > y_2$ and $(y_1, y_2, 0) \neq (5, 4, 0)$ we can easily observe that $|y_2-4| \leq \frac{6}{7}cdot5 \leq \frac{6}{7}M_T(x, y, u, v)$.

Thus $\psi(d(u,v)) = d(u,v) = y_2 + 4 \le \frac{6}{7}M_T(x,y,u,v) = M_T(x,y,u,v) - \frac{1}{7}M_T(x,y,u,v) = \psi(M_T(x,y,u,v)) - \phi(M_T(x,y,u,v))$.

Therefore in this case (3.1) is satisfied.

Therefore, in all the cases inequality (3.1)holds. Thus, we can conclude that T is a generalized $(\psi - \phi)-$ weak proximal contraction. Moreover all the hypotheses of Theorem 3.2 are satisfied. Hence T has a unique best proximity point.
Here we can not apply Corollary 3.6 to show that T has a best proximity point, since T does not satisfy the condition (3.18) of Corollary 3.6 at $x = (4,5,0), y = (5,4,0)$.

4. Consequences

In this section we introduce the following definition and obtain some results of best proximity points.

Definition 4.1. Let A and B be two nonempty subsets of a metric space (X,d) . A mapping $T : A \to B$ is said to $(\psi - \phi)$-weak proximal contraction type if there exists $\psi \in \Psi$ and $\phi \in \Phi$ such that for all $x, y, u, v \in A$,

$$\left.\begin{array}{l} \frac{1}{2}d(x,u) \le d(x,y) \\ d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies \psi(d(u,v)) \le \psi(d(x,y)) - \phi(d(x,y)).$$

Theorem 4.2. *Let A and B be two nonempty and closed subsets of a complete metric space (X,d). Let $T : A \to B$ be a continuous mapping. Suppose the following conditions are satisfied:*

i): *T is a $(\psi - \phi)-$ weak proximal contraction type mapping.*
ii): *$T(A_0) \subseteq B_0$.*
iii): *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A,B)$.*

Then T has a unique best proximity point.

Proof. Let $x_1, x_0 \in A$ such that $d(x_1, Tx_0) = d(A,B)$. As in the proof of Theorem 3.2 we construct a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A,B) \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

and converging to some $x^* \in A_0$.
Since T is a continuous mapping and the metric d is continuous , we have that

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = \lim_{n\to\infty} d(A,B) = d(A,B).$$

Hence T has a best proximity point.
The proof of uniqueness of this best proximity point is similar to that as in Theorem 3.2.
∎

If we take $\psi(t) = t$ in Theorem 4.2 we get the following corollary.

Corollary 4.3. *Let A and B be two nonempty and closed subsets of a complete metric space (X,d). Let $T : A \to B$ be a mapping such that for all $x, y, u, v \in A$ the following conditions are satisfied:*

i):

$$\left.\begin{array}{l}\frac{1}{2}d(x,u) \le d(x,y)\\ d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\} \implies d(u,v) \le d(x,y) - \phi(d(x,y)),$$

ii): $T(A_0) \subseteq B_0$.,
iii): *T has $RJ-$ property,*
iv): *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then T has a unique best proximity point.

Corollary 4.4. *Let A and B be two nonempty and closed subsets of a complete metric space (X, d). Let $T : A \to B$ be a continuous mapping such that for all $x, y, u, v \in A$ the following conditions are satisfied:*

i):

$$\left.\begin{array}{l} d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\} \implies \psi(d(u,v)) \le \psi(d(x,y)) - \phi(d(x,y)),$$

ii): $T(A_0) \subseteq B_0$,
iii): *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then T has a unique best proximity point.

If we take $\psi(t) = t$ and $\phi(t) = (1 - k)t$, where $0 \le k < 1$, in Corollary 4.4 we get the following corollary.

Corollary 4.5. *Let A and B be two nonempty and closed subsets of a complete metric space (X, d). Let $T : A \to B$ be a mapping such that for all $x, y, u, v \in A$ the following conditions are satisfied:*

i):

$$\left.\begin{array}{l} d(u,Tx)=d(A,B)\\ d(v,Ty)=d(A,B)\end{array}\right\} \implies d(u,v) \le kd(x,y),$$

ii): $T(A_0) \subseteq B_0$,
iii): *there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$.*

Then T has a unique best proximity point.

Remark 4.6. In the case of self mappings Corollary 4.5 reduces to the Banach contraction principle.

The following example shows the generalization of the main result (Theorem 3.2) of this paper as compared to Theorem 4.2.

Example 4.7. Let $X = R^3$, $d : X \times X \to R$ defined by

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$

$A = \{(1, 1, 0), (1, 4, 0), (4, 1, 0)\}$, $B = \{(1, 1, 1), (1, 4, 1), (4, 1, 1)\}$.
We define $T : A \to B$ by

$$T(x_1, x_2, 0) = \begin{cases} (1, 1, 1), & \text{if } x_1 \le x_2;\\ (1, 4, 1), & \text{if } x_1 > x_2. \end{cases}$$

We can easily seen that $d(A,B) = 1$. For $x = (1,1,0)$, $y = (4,1,0)$, $u = (u_1, u_2, 0)$ and $v = (v_1, v_2, 0)$. From $d(u, Tx) = 1$ we get $u = (1,1,0)$ and from $d(v, Ty) = 1$ we get $v = (1,4,0)$. Since there is no $\psi \in \Psi$ and $\phi \in \Phi$

$$\frac{1}{2}d(x,u) = \frac{1}{2}d((1,1,0),(1,1,0)) = 0 \leq d((1,1,0),(4,1,0) = 3$$

implies $\psi(d(u,v)) = \psi(3) \leq \psi(3) - \phi(3) = \psi(d(x,y)) - \phi(d(x,y))$.
Thus we can not apply Theorem 4.2 to conclude that T has a best proximity point. However, all the hypotheses of Theorem 3.2 of this paper can easily be verified for the map T and conclude that it has a unique best proximity point.

5. Application in Fixed point theory

Here we deduce certain new and general fixed point results for Suzuki contractions and we prove the fixed point theorem which is proved by Shyam et al. [13] as follows. Our results contain properly the main theorem due to Suzuki [15] and many of its extensions [14].

Theorem 5.1. *Let X be a complete metric space and $T : X \to X$ be such that for every $x, y \in X$,*

$$\frac{1}{2}d(x;Tx) \leq d(x,y) \text{ implies } \psi(d(Tx,Ty)) \leq \psi(M_g(Tx,Ty)) - \phi(M_g(Tx,Ty)),$$

where $\psi \in \Psi$ and $\phi \in \Phi$ are defined as in Theorem 3.2. Then T has a unique fixed point.

Proof. Let $A = B = X$ in Theorem 3.2. Clearly $A_0 = X = B_0$. Thus, $T(A_0) \subseteq B_0$ and any self map has $RJ-$ property. we prove that T is a generalized (ψ,ϕ)-weak proximal contraction type map. Let $x, y, u, v \in X$, satisfying the following conditions

$$\begin{cases} \frac{1}{2}d(x,u) \leq d(x,y), \\ d(u,Tx) = d(A,B), \\ d(v,Ty) = d(A,B). \end{cases}$$

Since $d(A,B) = 0$, we have $u = Tx$ and $v = Ty$. Then by hypothesis of thm, we have $\psi(d(u,v)) = \psi(d(Tx,Ty)) \leq \psi(M_g(Tx,Ty)) - \phi(M_g(Tx,Ty))$,
where

$$\begin{aligned} M_g(Tx,Ty) &= \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\} \\ &= \max\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\} \\ &= M_T(x,y,u,v). \end{aligned}$$

Therefore

$$\psi(d(u,v)) \leq \psi(M_T(x,y,u,v)) - \phi(M_T(x,y,u,v)),$$

which implies that T is a generalized $(\psi - \phi)$-weak proximal contraction type map.
Further condition (iv) of Theorem 3.2 is satisfied by taking any arbitrary $x_0 \in X$, we have that $d(Tx_0, Tx_0) = d(A,B) = 0$. Let $x_1 = Tx_0$, so that there exist x_0, x_1 in A such that $d(x_1, Tx_0) = d(A,B)$.
Therefore all the conditions of Theorem 3.2 are satisfied. Consequently there exists a unique $x^* \in X$ such that $d(x^*, Tx^*) = 0$. This implies $x^* = Tx^*$.
Hence T has a unique fixed point. ∎

References

[1] G. V. R. Babu and P. D. Sailaja, *A fixed point theorem of generalized weak contractive maps in orbitally complete metric spaces*, **Thai Journal of Mathematics 9**(1)(2011), 1-10.

[2] Fan, K. *Extensions of two fixed point theorems of F.E. Browder*, **Math.Z. 112**(1969), 234–240.

[3] M. Gabeleh, *Best proximity point theorems via proximal non-self mappings*, **J. Optim. Theory Appl. 164**(2)(2015), 565–576.

[4] J. Hamzehnejadi, R. Lashkaripour, *Best proximity points for generalized $\alpha-\phi-$ Geraghty proximal contraction mappings and its applications*, **Fixed point theory and Applications 2016**.

[5] N. Hussain, M. Hezarjaribi, MA.Kutbi, P. Salimi, *Best proximity results for Suzuki and convex type contractions*, **Fixed point theory and App.**, **2016**(1), 14 pages.

[6] M. Jleli and B. Samet, *Best proximity points for $\alpha-\psi$-proximal contractive type mappings and applications*, **Bulletin des Sciences Mathematiques**, **137**(8)(2013), 977-995.

[7] J. B. Prolla, *Fixed-point theorems for set-valued mappings and existence of best approximants* **Numerical Functional Analysis and Optimization**, **5**(4)(1982-1983), 449-455.

[8] V. S. Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, **Nonlinear Analysis: Theory, Methods and Applications**, **74**(14)(2011), 4804–4808.

[9] S. Reich, Approximate selections, *best approximations, fixed points, and invariant sets*, **Journal of Mathematical Analysis and Applications 62**(1)(1978), 104-113.

[10] B. Samet, *Some results on best proximity points*, **Journal of Optimization Theory and Applications**, **159**(1)(2013), 281-291.

[11] K.K.M Sarma, Yohannes G., *Best Proximity Points for generalized $(\alpha-\eta-\psi)$-Geraghty Proximal Contraction mappings*, **Mathematica Moravica**, 21(2)(2017), 85–102.

[12] V.M. Sehgal and S. P. Singh, *A generalization to multi functions of Fans best approximation theorem*, **Proceedings of the American Mathematical Society**, **102**(3)(1988), 534-537.

[13] Shyam Lal Singh, Raj Kamal, M. De la Sen, Renu Chugh, *A Fixed Point Theorem for Generalized Weak Contractions*, **Filomat**, **297**(2015), 1481–1490.

[14] T. Suzuki, *A new type of fixed point theorem in metric spaces.* **Nonlinear Anal. 71**(11)(2009), 5313–5317.

[15] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness.*, **Proc. Am. Math. Soc. 136**(2008), 1861–1869.

Bangmod International
Journal of Mathematical Computational Science
ISSN: 2408-154X

Journal office:
Theoretical and Computational Science Center (TaCS)
Science Laboratory Building, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, Thailand 10140
Website: http://tacs.kmutt.ac.th/
Email: tacs@kmutt.ac.th